Stat 401 - 9/27/17

TA Info: Mr. Shuai Hao
Office Hrs: Wednesday 12-3pm in SEO 430.

Sec 1.10 - Important Inequalities.

Thm If $\mathbb{E}[X^m]$ exists $(< \infty)$, then $\mathbb{E}[X^k]$ exists for all $0 \le k \le m$.

Proof on pg 68 of text.

Thm Markov's Inequality

Let X be a RV.
Let $g(X)$ be a function of X such that $g(X) \geq 0$

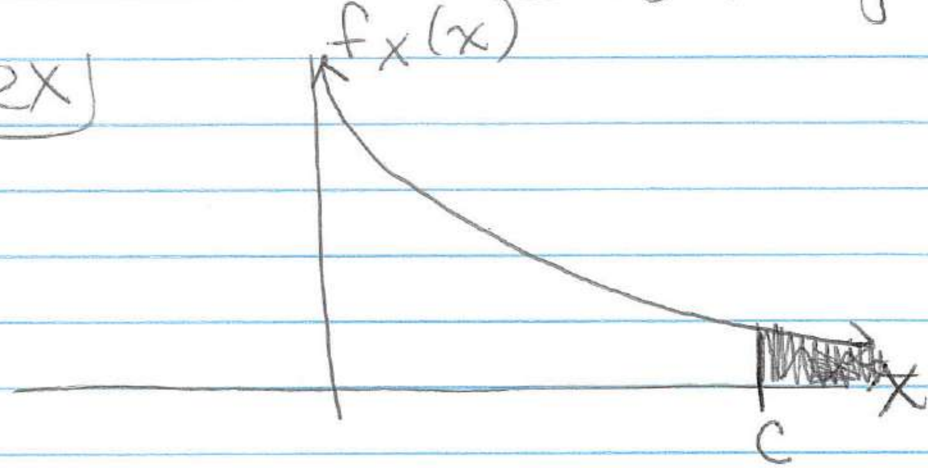
Let c be a positive constant.
Assume $\mathbb{E}[g(X)]$ exists (i.e. $< \infty$)
Then

$$\mathbb{P}[g(X) \geq c] \leq \frac{\mathbb{E}[g(X)]}{c}$$

Remarks / Notes:

- Any function of a RV is still a RV.
- Ineq. give a "tail estimate" for X.
- Useful when c is large.

ex)



- If we have a finite expectation, $\mathbb{E}[X] < \infty$ [$g(X) = X$ here]. then $\mathbb{P}[X \geq c]$ cannot be too big

ex) Suppose X is a RV with pdf

$$f_X(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & \text{o.w.} \end{cases}$$

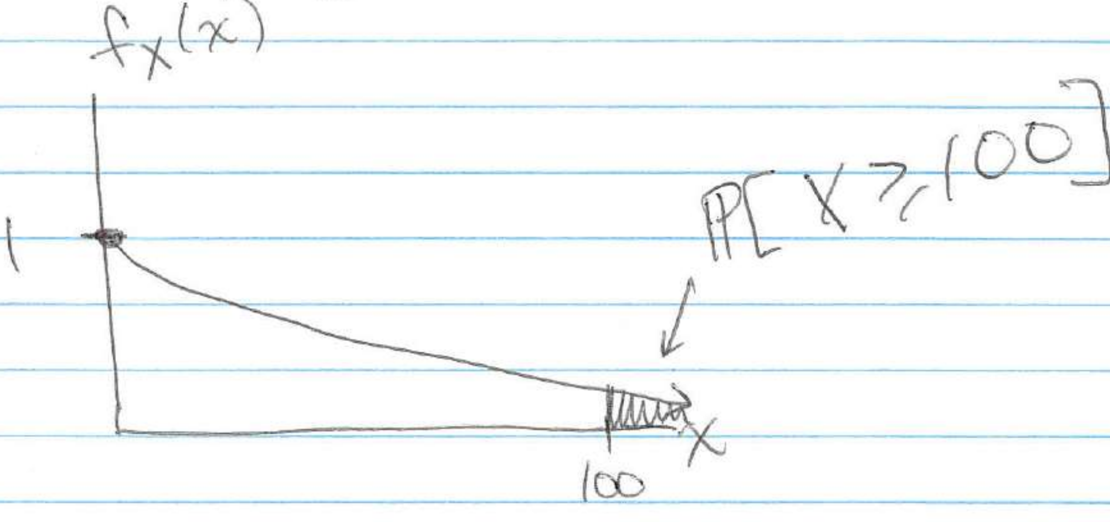
Find upper bound for $\mathbb{P}[X \geq 100]$

$\boxed{\text{SOL}}$ $g(X) = X$

By Markov's Ineq.

$$\int_{100}^{\infty} e^{-x}\,dx = \mathbb{P}[X \geq 100] \leq \frac{\mathbb{E}[X]}{100} = \frac{1}{100}$$

Previously found $\mathbb{E}[X] = 1$.



Thm] Chebyshev's Inequality

Let X be a RV with finite variance σ^2. For any $c > 0$

$$\mathbb{P}[|X - \mu| \geq c] \leq \frac{\mathrm{Var}[X]}{c^2}$$

μ: the mean $\mathbb{E}[x]$

Proof

$$\mathbb{P}[\,|X-\mu| \geq c\,] = \mathbb{P}[\,|X-\mu|^2 \geq c^2\,]$$

$$g(X) = |X-\mu|^2 \qquad \leq \frac{\mathbb{E}[\,|X-\mu|^2\,]}{c^2}$$

by Markov.

$$= \frac{\mathbb{E}[\,(X-\mu)^2\,]}{c^2}$$

$$= \frac{\text{var}[X]}{c^2}$$

ex Let X be a positive RV. Find an upper bound for $\mathbb{P}[X \geq 5\mu]$.

Positive RV: $\mathbb{P}[X > 0] = 1$

$$\mathbb{P}[X \geq 5\mu] = \mathbb{P}\left[\frac{X}{\mu} \geq 5\right]$$

exact: Assume continuous $\int_{5\mu}^{\infty} f_X(x)\,dx$

$$g(X) = \frac{X}{\mu} \qquad c = 5$$

$$\mathbb{P}\left[\frac{X}{\mu} \geq 5\right] \leq \frac{\mathbb{E}\left[\frac{X}{\mu}\right]}{5} \quad \text{Markov.}$$

$$\mathbb{E}\left[\frac{X}{\mu}\right] = \frac{1}{\mu}\mathbb{E}[X] = \frac{1}{\mathbb{E}[X]} \cdot \mathbb{E}[X]$$

$$= 1$$

$$= \frac{1}{5}$$

B/c X was pos RV, then

$$\mathbb{E}[X] = \mu > 0$$

cont. case $\int_{-\infty}^{\infty} x \cdot f_X(x)\,dx.$

pos# pos.

In general:

$$\mathbb{P}[X \geq A\mu] \leq \frac{1}{A} \quad \text{for any } A > 0$$

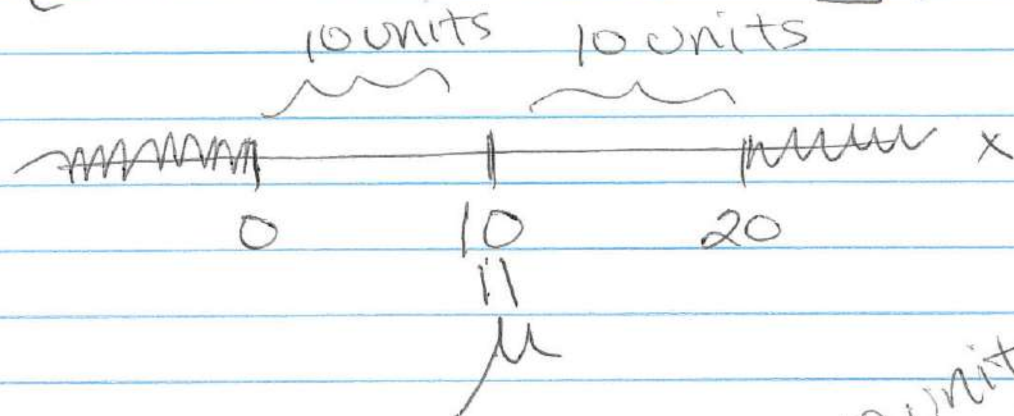
pos. RV.

ex) Let X be a RV with $\mu = 10$ and $\sigma^2 = 40$.

Find an upper bound for

$$\mathbb{P}[X \geq 20 \text{ or } X \leq 0].$$

SOL



10 units away

$$\mathbb{P}[X \geq 20 \text{ or } X \leq 0]$$

$$= \mathbb{P}\left[\,|X - \underset{\mu}{10}| \geq \underset{c}{10}\,\right]$$

$$\leq \frac{\mathrm{Var}[X]}{10^2} = \frac{40}{100} = 0.4$$

$\{|x|<5\} = \{-5<x<5\}$.

ex) Let X be a RV with $\mu = 3$ and $Var[X] = 10$.

Use Chebychev to get a lower bound for

$$\mathbb{P}[-2 < X < 8].$$

Sol

$$\mathbb{P}[-2 < X < 8] = \mathbb{P}[-2-\mu < X-\mu < 8-\mu]$$

$$= \mathbb{P}[-2-3 < X-\mu < 8-3]$$

$$= \mathbb{P}[-5 < X-\mu < 5]$$

$$= \mathbb{P}[|X-\mu| < 5]$$

$$= 1 - \mathbb{P}[|X-\mu| \geq 5]$$

Aside

Complement Rule.

$$\mathbb{P}[|X-\mu| \geq 5] \leq \frac{Var[X]}{5^2} = \frac{10}{25} = \frac{2}{5}$$

$$\Rightarrow -\mathbb{P}[|X-\mu| \geq 5] \geq -\frac{2}{5}$$

$$\geq 1 - \frac{2}{5} = \frac{3}{5}$$

thm) Jensen's inequality.

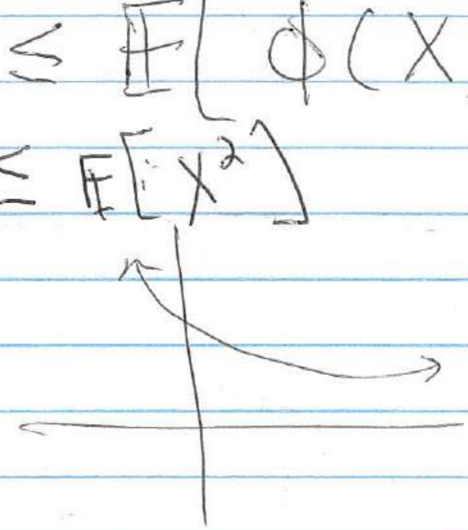
Suppose ϕ is a convex function on an open interval I, and X is a RV whose support is contained in I and has finite expectation.

Then

$$\phi[\mathbb{E}(X)] \leq \mathbb{E}[\phi(X)]$$

If $\phi(x)=x^2$ $\quad [\mathbb{E}(X)]^2 \leq \mathbb{E}[X^2]$

convex example



ex) $\mathbb{E}[X^2] \geq [\mathbb{E}(X)]^2$

$$\phi(x) = x^2$$

Why important?

$$V(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$\geq 0$$