Stat 401 - 10/2/17

HW - Study for Exam on Friday.

Sec 2.1 Continued

Continuous Random Vector

Let $\mathcal{D}$ be our space.
Let $\vec{X}$ is a RV with continuous CDF.

$$\vec{X} = (X_1, X_2)$$

Let $f_{X_1,X_2}(x_1, x_2)$ be the joint pdf.

$$\Updownarrow$$
$$f_{1,2}(x_1, x_2)$$

Define its CDF as.

$$F_{X_1,X_2}(x_1, x_2) = \mathbb{P}[X_1 \le x_1, X_2 \le x_2]$$
$$= \int_{-\infty}^{x_2}\int_{-\infty}^{x_1} f_{1,2}(x_1, x_2)\,dx_1\,dx_2$$

for all $(x_1, x_2) \in \mathbb{R}^2$.

Calculate joint pdf as

$$f_{X_1,X_2}(x_1, x_2) = \frac{\partial^2 F_{X_1,X_2}(x_1, x_2)}{\partial x_1 \partial x_2}$$

Properties.

1. $f_{X_1,X_2}(x_1,x_2) \geq 0$

2. $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\,dx_1\,dx_2 = 1$$

3. $$\mathbb{P}\left[(X_1,X_2)\in A\right] = \iint_A f_{1,2}(x_1,x_2)\,dx_1\,dx_2$$

for $A \subseteq D$

$0 \le x_1 \le 1$
$0 \le x_2 \le 1$

ex) $$f_{1,2}(x_1,x_2) = \begin{cases} 6x_1^2 x_2, & 0 \le x_1, x_2 \le 1 \\ 0, & \text{o.w.} \end{cases}$$

Find
$$\mathbb{P}\left[0 < X_1 < \tfrac{3}{4},\ \tfrac{1}{3} < X_2 < 2\right]$$

$$= \int_{\frac{1}{3}}^{2}\int_{0}^{\frac{3}{4}} f_{1,2}(x_1,x_2)\,dx_1\,dx_2$$

$$= \int_{\frac{1}{3}}^{1}\int_{0}^{\frac{3}{4}} 6x_1^2 x_2\,dx_1\,dx_2 + \int_{1}^{2}\int_{0}^{\frac{3}{4}} 0\,dx_1\,dx_2$$

$$= \frac{3}{8}$$

we can look at X_1 and X_2 individually. 3

Want CDF or PDF of X_1

$$F_{X_1}(x_1) = \mathbb{P}[X_1 \le x_1]$$
$$= \int_{-\infty}^{x_1}\int_{-\infty}^{\infty} f_{1,2}(x_1, x_2)\, dx_2\, dx_1$$

ex) (X, Y)

$$f(x,y) = \begin{cases} e^{-x-y}, & x>0,\ y>0 \\ 0, & \text{o.w.} \end{cases}$$

Find CDF of X.

$$F_X(x) = \mathbb{P}[X \le x] = \mathbb{P}[X \le x, -\infty < Y < \infty]$$
$$= \int_0^x \int_0^\infty e^{-w-y}\, dy\, dw$$
$$= \int_0^x -e^{-w-y}\Big|_0^\infty dw$$
$$= \int_0^{x} e^{-w}\, dw = 1 - e^{-x} \quad \boxed{\text{for } x>0}$$

Find PDF of X:

Find $\frac{d F_X(x)}{dx} = f_X(x)$.

$$= \frac{d}{dx}\left(1-e^{-x}\right)$$

$$= e^{-x} \text{ for } x>0$$

PDF $\boxed{f_X(x) = \begin{cases} e^{-x}, & x>0 \\ 0, & \text{o.w.} \end{cases}}$

Marginal Distribution.

PDF of X_1

$$f_{X_1}(x_1) = \frac{d F_{X_1}(x_1)}{dx_1} = \int_{-\infty}^{\infty} f(x_1, x_2)\,dx_2$$

PDF of X_2

$$f_{X_2}(x_2) = \frac{d F_{X_2}(x_2)}{dx_2} = \int_{-\infty}^{\infty} f(x_1, x_2)\,dx_1$$

ex) X_1, X_2 have joint density

$$f(x_1, x_2) = \begin{cases} x_1 + x_2, & 0 < x_1 < 1,\ 0 < x_2 < 1 \ \ast \\ 0, & o.w. \end{cases}$$

a) Find marg. dist. of X_1.

$$\int_0^1 (x_1 + x_2)\, dx_2 = x_1 + \tfrac{1}{2} \text{ for } 0 < x_1 < 1$$

$$f_{X_1}(x_1) = \begin{cases} x_1 + \frac{1}{2}, & 0 < x_1 < 1 \\ 0, & o.w. \end{cases}$$

b) Find marg dist. of X_2.

$$\int_0^1 (x_1 + x_2)\, dx_1 = x_2 + \tfrac{1}{2} \text{ for } 0 < x_2 < 1$$

$$f_{X_2}(x_2) = \begin{cases} x_2 + \frac{1}{2}, & 0 < x_2 < 1 \\ 0, & o.w. \end{cases}$$

c) Find $\mathbb{P}\left[X_1 \le \frac{1}{2}\right]$.

Opt 1: use marg. dist

$$\mathbb{P}\left[X_1 \le \frac{1}{2}\right] = \int_0^{\frac{1}{2}} \left(x_1 + \frac{1}{2}\right) dx_1$$

$$= \frac{3}{8}$$

Opt 2: use joint pdf

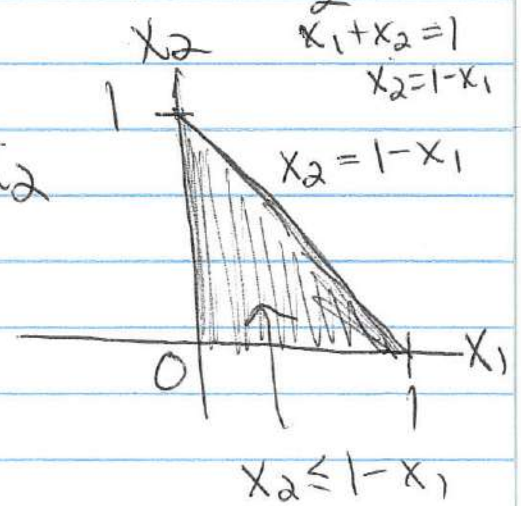
$$\mathbb{P}\left[X_1 \le \frac{1}{2}\right] = \int_0^1 \int_0^{\frac{1}{2}} (x_1 + x_2)\, dx_1\, dx_2$$

$$= \frac{3}{8}$$

d) $\mathbb{P}\left[X_1 + X_2 \le 1\right]$

$0 < x_1 \le 1 - x_2$

$$= \int_0^1 \int_0^{1-x_2} (x_1 + x_2)\, dx_1\, dx_2$$



$$= \frac{1}{3}$$

Expectations

Let $\vec{X} = (X_1, X_2)$

Find $\mathbb{E}\left[g(X_1, X_2) \right]$

$$= \begin{cases} \displaystyle\sum_{x_1 \in D_{X_1}} \sum_{x_2 \in D_{X_2}} g(x_1, x_2)\, p_{1,2}(x_1, x_2) & \text{if discrete} \\ \displaystyle\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x_1, x_2)\, f_{1,2}(x_1, x_2)\, dx_1\, dx_2 & \text{if continuous.} \end{cases}$$

Property:

If $Y_1 = g_1(X_1, X_2)$ and

$Y_2 = g_2(X_1, X_2)$

where Y_1, Y_2 are RVs

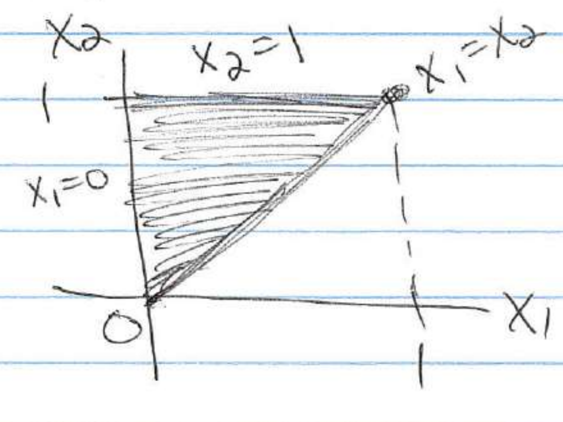
Then for $k_1, k_2, k_3 \in \mathbb{R}$

$$\mathbb{E}\left[k_1 Y_1 + k_2 Y_2 + k_3 \right] = k_1 \mathbb{E}[Y_1] + k_2 \mathbb{E}[Y_2] + k_3$$

ex) $f_{1,2}(x_1, x_2) = \begin{cases} 8x_1x_2, & 0 < x_1 < x_2 < 1 \\ 0, & \text{o.w.} \end{cases}$

Find $\mathbb{E}[\underbrace{X_1 X_2^2}_{g(X_1,X_2)}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2^2 \cdot f(x_1, x_2)\, dx_1 dx_2$

$= \int_0^1 \int_0^{x_2} x_1 x_2^2 (8x_1x_2)\, dx_1 dx_2$



$= \dfrac{8}{21}$