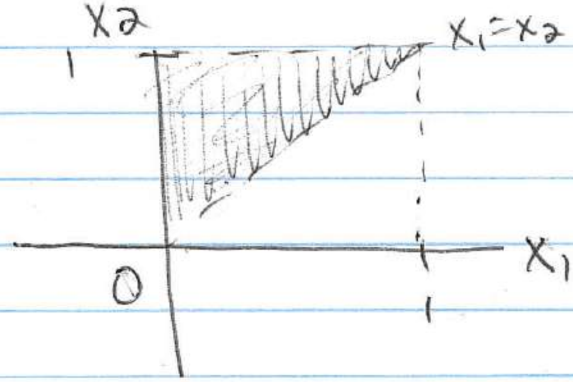
Stat 401 - 10/9/17 HW due next Mon. Problems on Blackboard.

Sec 2.1 - Continued.

ex) $$f_{1,2}(x_1, x_2) = \begin{cases} 8x_1x_2, & 0 < x_1 < x_2 < 1 \\ 0, & \text{o.w.} \end{cases}$$

$$\mathbb{E}[X_1X_2^2] = \int_0^1 \int_0^{x_2} 8x_1^2 x_2^3 \, dx_1 \, dx_2 = \frac{8}{21}$$



$$\int_0^{x_2} 8x_1x_2 \, dx_1$$
$$\frac{8x_1^2}{2} x_2 \Big|_0^{x_2}$$
$$= 4x_2^2 x_2$$
$$= 4x_2^3$$

Find $\mathbb{E}[X_2^2]$.

$$\mathbb{E}[X_2^2] = \int_0^1 \int_0^{x_2} x_2^2 \cdot 8x_1x_2 \, dx_1 \, dx_2$$

Alternate: Find marginal dist of X_2.

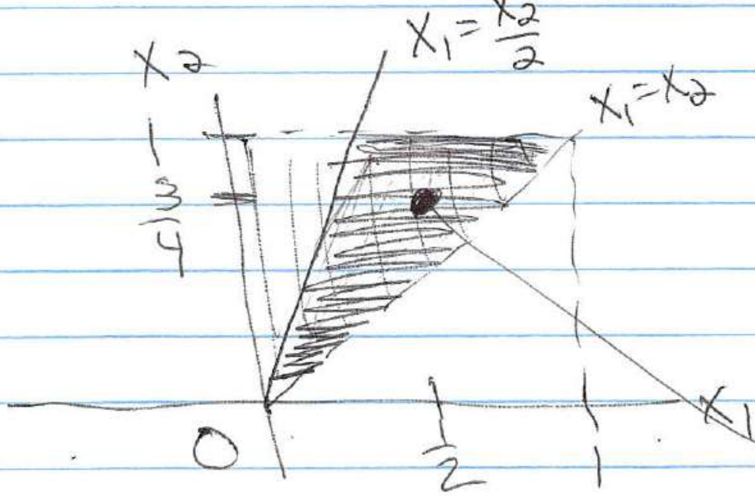
$$f_2(x_2) = \begin{cases} \int_0^{x_2} 8x_1x_2 \, dx_1 = 4x_2^3, & 0 < x_2 < 1 \\ 0, & \text{o.w.} \end{cases}$$

$$E[X_2^2] = \int_0^1 x_2^2 \cdot 4 x_2^3 \, dx_2$$

$$E[7X_1X_2^2 + 5X_2^2]$$

$$= 7E[X_1X_2^2] + 5E[X_2^2]$$

$$P\left[X_1 > \frac{X_2}{2}\right] = \iint 8x_1x_2\,dx_1\,dx_2$$



$\left[X_1 > \frac{X_2}{2}\right]$

$$\int_0^1 \int_{\frac{x_2}{2}}^{x_2} 8x_1x_2\,dx_1\,dx_2$$

$x_1 = \frac{1}{2}$ $x_2 = \frac{3}{4}$

$\frac{x_2}{2} = \frac{3}{8}$

$x_1 > \frac{x_2}{2}$

$0 < x_1 < x_2 < 1$

$0 < \frac{x_2}{2} < x_1 < x_2 < 1$

$2x_1 > x_2$

$$= \int_0^{\frac{1}{2}} \int_{x_1}^{2x_1} 8x_1x_2\,dx_2\,dx_1 + \int_{\frac{1}{2}}^{1} \int_{x_1}^{1} 8x_1x_2\,dx_2\,dx_1$$

def) MGF of Random Vectors.

Let (X_1, X_2) be a random vector.

The MGF of (X_1, X_2) is

$$M_{X_1,X_2}(t_1, t_2) = \mathbb{E}\underbrace{\left[e^{t_1X_1+t_2X_2}\right]}_{\text{needs to exist and be finite.}}$$

Marginal MGF of X_1 is

$$M_{X_1,X_2}(t_1, 0) = \mathbb{E}\left[e^{t_1X_1}\right]$$

Marginal MGF of X_2 is

$$M_{X_1,X_2}(0, t_2) = \mathbb{E}\left[e^{t_2X_2}\right]$$

Sec 2.2 Transformations (Bivariate RVs).

Setup:
- (X_1, X_2) is a random vector.
- $g(X_1, X_2)$ is a function

set $Y = g(X_1, X_2)$.

Y is a RV

Goals:

① If (X_1, X_2) is discrete, find pmf of Y.

② If (X_1, X_2) is continuous, find CDF and pdf of Y.

ex) Discrete RV (X_1, X_2)

$$p_{1,2}(x_1, x_2) = \frac{\mu_1^{x_1} \mu_2^{x_2} e^{-\mu_1} e^{-\mu_2}}{x_1! \, x_2!}$$

for $x_1 = 0, 1, 2, \ldots$
$x_2 = 0, 1, 2, \ldots$

μ_1, μ_2 are positive constants.
Find pmf of $Y = g(X_1, X_2) = X_1 + X_2$

SOL) $D_Y = \{0, 1, 2, \dots\}$

PMF for Y

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$$p_Y(y) = \mathbb{P}[Y = y] = \mathbb{P}[X_1 + X_2 = y]$$

$$= \sum_{x_1 + x_2 = y} p_{1,2}(x_1, x_2)$$

$$= \sum_{x_1 + x_2 = y} \frac{\mu_1^{x_1} \mu_2^{x_2} e^{-\mu_1} e^{-\mu_2}}{x_1! \, x_2!}$$

use substitution. Let $x_2 = y - x_1$

$$= \sum_{x_1 = 0}^{x_1 + x_2} \frac{\mu_1^{x_1} \mu_2^{y - x_1} e^{-\mu_1} e^{-\mu_2}}{x_1! \, (y - x_1)!}$$

$$= \frac{e^{-\mu_1} e^{-\mu_2}}{y!} \sum_{x_1 = 0}^{x_1 + x_2} \frac{\mu_1^{x_1} \mu_2^{y - x_1} \, y!}{x_1! \, (y - x_1)!}$$

$$= \frac{e^{-\mu_1} e^{-\mu_2}}{y!} \underbrace{\sum_{x_1 = 0}^{x_1 + x_2 = y} \binom{y}{x_1} \mu_1^{x_1} \mu_2^{y - x_1}}_{\text{Binomial Expansion.}}$$

$$\sum_{k=0}^{n} a^k b^{n-k} \binom{n}{k} = (a+b)^n$$

$$= \frac{e^{-\mu_1} e^{-\mu_2}}{y!} (\mu_1 + \mu_2)^y$$

for $y = 0, 1, 2, \ldots$

General Technique for Discrete Transformations

(change of variable)

Let $y_1 = u_1(x_1, x_2)$
$y_2 = u_2(x_1, x_2)$ ← 1-1 transformations

functions.

Solve for x_1 and x_2.

$x_1 = w_1(y_1, y_2)$
$x_2 = w_2(y_1, y_2)$

functions.

Then for a discrete Random Vector, we have

$$P_{Y_1, Y_2}(y_1, y_2) = P_{X_1, X_2}\big(w_1(y_1, y_2), w_2(y_1, y_2)\big)$$

pmf for (Y_1, Y_2)

ex) Discrete (X_1, X_2).

same pmf

we want $Y = X_1 + X_2$

(*) Define $Y_1 = X_1 + X_2$

$Y_2 = X_2$

(*) Solve for x_1, x_2.

$x_1 = 0, 1, 2, \dots$
$x_2 = 0, 1, 2, \dots$

$x_2 = y_2$

$x_1 = y_1 - x_2 = y_1 - y_2$

(*) Spaces.

$D_{Y_1} = \{0, 1, 2, \dots\}$

$D_{Y_2} = \{0, 1, 2, \dots, y_1\}$.

(*) Joint pmf

$$p_{Y_1, Y_2}(y_1, y_2) = \frac{\mu_1^{y_1 - y_2} \mu_2^{y_2} e^{-\mu_1} e^{-\mu_2}}{(y_1 - y_2)! \, y_2!}$$

(4) Find marginal dist wrt Y_1

$$P_{Y_1}(y_1) = \sum_{y_2=0}^{y_1} \frac{\mu_1^{y_1-y_2}\,\mu_2^{y_2}\,e^{-\mu_1}\,e^{-\mu_2}\,y_1!}{(y_1-y_2)!\,y_2!\,y_1!}$$

$$= \frac{e^{-\mu_1}\,e^{-\mu_2}}{y_1!} \sum_{y_2=0}^{y_1} \frac{\mu_1^{y_1-y_2}\,\mu_2^{y_2}\,y_1!}{(y_1-y_2)!\,y_2!}$$

$$= \binom{y_1}{y_2}$$

$$= \frac{e^{-\mu_1}\,e^{-\mu_2}}{y_1!}(\mu_2+\mu_1)^{y_1}$$