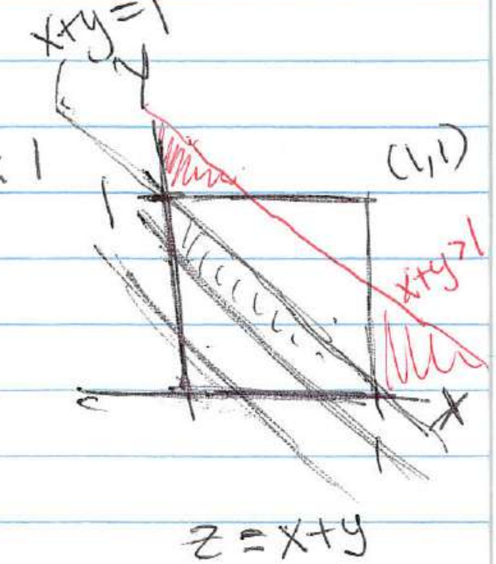
Sec 2.2 - Continued.

ex| Continuous Random Vector Transformation (Distribution Function Method)

A person chooses a point (X, Y) at random from the unit square.



$$f_{X,Y}(x,y) = \begin{cases} 1, & 0 < x, y < 1 \\ 0, & \text{o.w.} \end{cases}$$

We are interested in $Z = X + Y$

Note: $0 < z < 2$

⊛ Find CDF of Z.

$$F_Z(z) = \mathbb{P}[Z \le z] = \underbrace{\mathbb{P}[X + Y \le z]}_{\text{depends on } z\text{-value.}}$$

If $0 < z < 1$ we have a different value than if $1 \le z < 2$

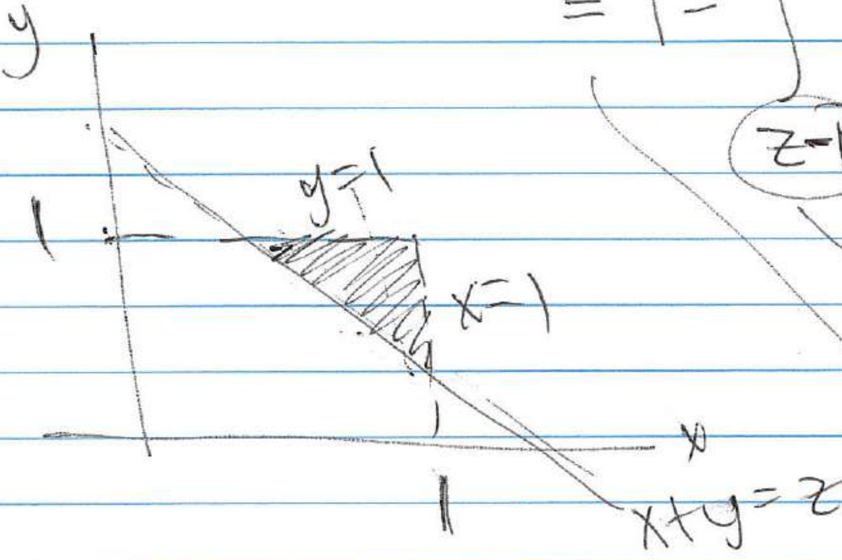
⊛ Case 1: $0 < z < 1$

$$\mathbb{P}[\underbrace{X + Y \le z}_{0 < y \le z - x}] = \int_0^z \int_0^{z-x} 1 \, dy\,dx = \frac{z^2}{2}$$

Case 2: $1 \le z < 2$

$$\mathbb{P}[X+Y \le z] = 1 - \mathbb{P}[X+Y \ge z]$$

$$= 1 - \int_{z-1}^{1}\int_{z-x}^{1} 1 \, dy\, dx$$



$z-1 \to x \ge z - y$

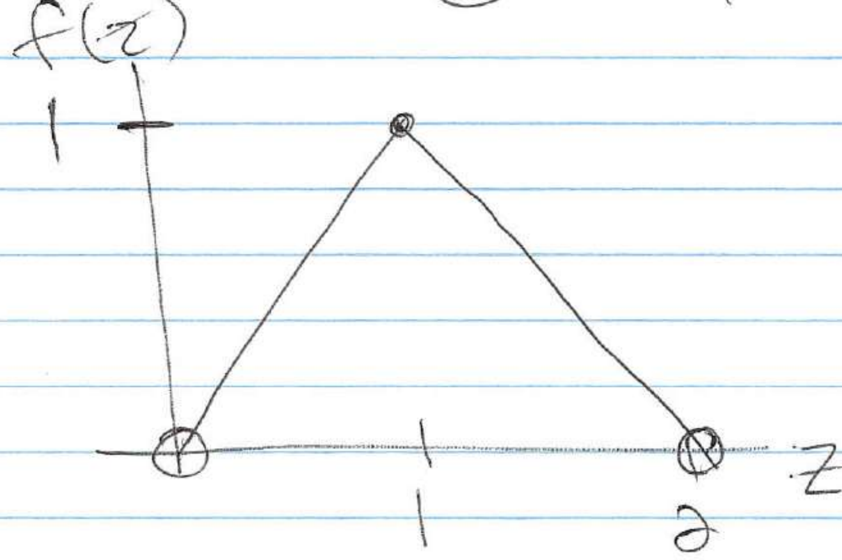
$> z - 1$

$$= 1 - \frac{(2-z)^2}{2}$$

$$F_Z(z) = \begin{cases} 0, & z < 0 \\ \frac{z^2}{2}, & 0 \le z < 1 \\ 1 - \frac{(2-z)^2}{2}, & 1 \le z < 2 \\ 1, & 2 \le z \end{cases}$$

pdf for Z:

$$f_Z(z) = \begin{cases} z, & 0 < z < 1 \\ 2 - z, & 1 \le z < 2 \\ 0, & \text{o.w.} \end{cases}$$



<u>General Technique (Change of Variable)</u>

Let (X_1, X_2) be a continuous random vector with joint pdf $f_{1,2}(x_1, x_2)$

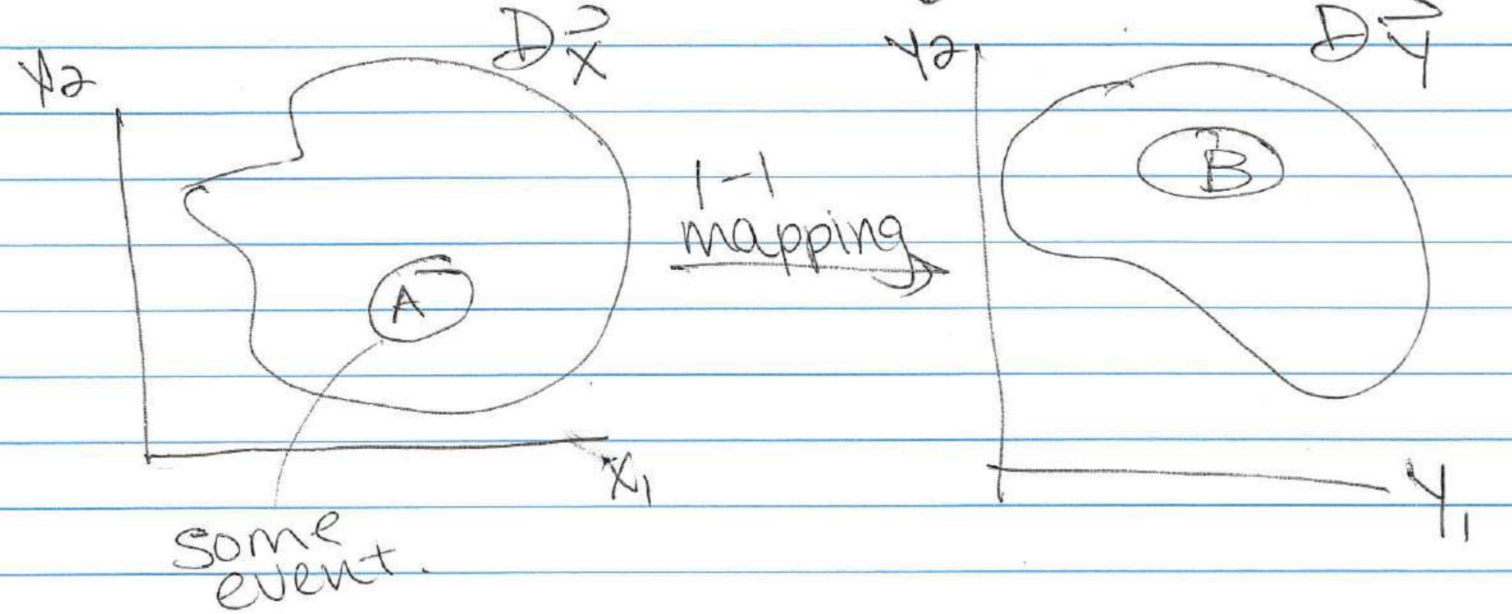
Let $Y_1 = u_1(X_1, X_2)$
$Y_2 = u_2(X_1, X_2)$
functions

one-to-one transforms.

Goal: Find $f_{Y_1,Y_2}(y_1,y_2)$

OR

$$\mathbb{P}\left[(Y_1,Y_2)\in B\right] = \iint_B f_{Y_1,Y_2}(y_1,y_2)\,dy_1\,dy_2$$



$$\mathbb{P}\left[(Y_1,Y_2)\in B\right] = \mathbb{P}\left[(X_1,X_2)\in A\right]$$
$$= \iint_A f_{1,2}(x_1,x_2)\,dx_1\,dx_2$$

How to change?

Use a change of variable functions

$$x_1 = w_1(y_1,y_2)$$
$$x_2 = w_2(y_1,y_2)$$

Need the Jacobian of the transformation.

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$ ← determinant.

Then

$$\iint_A f_{1,2}(x_1, x_2)\,dx_1\,dx_2$$

$$= \iint_B \underbrace{f_{1,2}\Big(w_1(y_1,y_2),\, w_2(y_1,y_2)\Big)\,|J|}_{\substack{\text{joint pdf for} \\ (Y_1,\, Y_2) \\ \text{where} \\ (Y_1, Y_2) \in \mathcal{D}_{\vec{Y}}}}\,dy_1\,dy_2$$

absolute value

make sure to identify this!

ex) Suppose (X_1, X_2) has joint pdf

$$f_{1,2}(x_1, x_2) = \begin{cases} 1, & 0 < x_1 < 1,\ 0 < x_2 < 1 \\ 0, & \text{o.w.} \end{cases}$$

$$D_{\vec{X}} = \{(x_1, x_2) : 0 < x_1 < 1,\ 0 < x_2 < 1\}$$

Suppose $Y_1 = X_1 + X_2 \Rightarrow y_1 = x_1 + x_2$

Suppose. $Y_2 = X_1 - X_2 \Rightarrow y_2 = x_1 - x_2$

$+$

This is 1-1.

$$\frac{y_1 + y_2}{2} = \frac{2x_1}{2}$$

(*) Solve for x_1 and x_2.

$$y_1 - y_2 = 2x_2$$

$$x_2 - -x_2$$

$$x_1 = w_1(y_1, y_2) = \tfrac{1}{2}(y_1 + y_2)$$

$$x_2 = w_2(y_1, y_2) = \tfrac{1}{2}(y_1 - y_2)$$

* Find J.

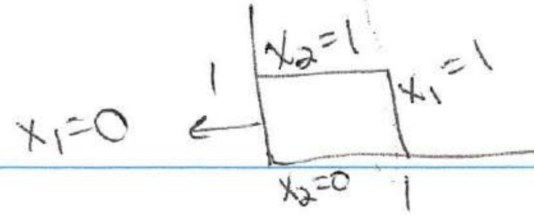
$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} = \frac{1}{2} & \frac{\partial x_1}{\partial y_2} = \frac{1}{2} \\ \frac{\partial x_2}{\partial y_1} = \frac{1}{2} & \frac{\partial x_2}{\partial y_2} = -\frac{1}{2} \end{vmatrix}$$

$$= \frac{1}{2}\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

* $|J| = \left|-\frac{1}{2}\right| = \frac{1}{2}$

* PDF of (Y_1, Y_2)

$$f_{1,2}(y_1, y_2) = \begin{cases} f_{X_1,X_2}\left(\frac{1}{2}(y_1+y_2), \frac{1}{2}(y_1-y_2)\right) \cdot |J| \\ \quad = 1\left(\frac{1}{2}\right) = \frac{1}{2}, & (Y_1, Y_2) \in D_{\vec{Y}} \\ 0, & \text{o.w.} \end{cases}$$



(4) Find $D_{\vec{Y}}$

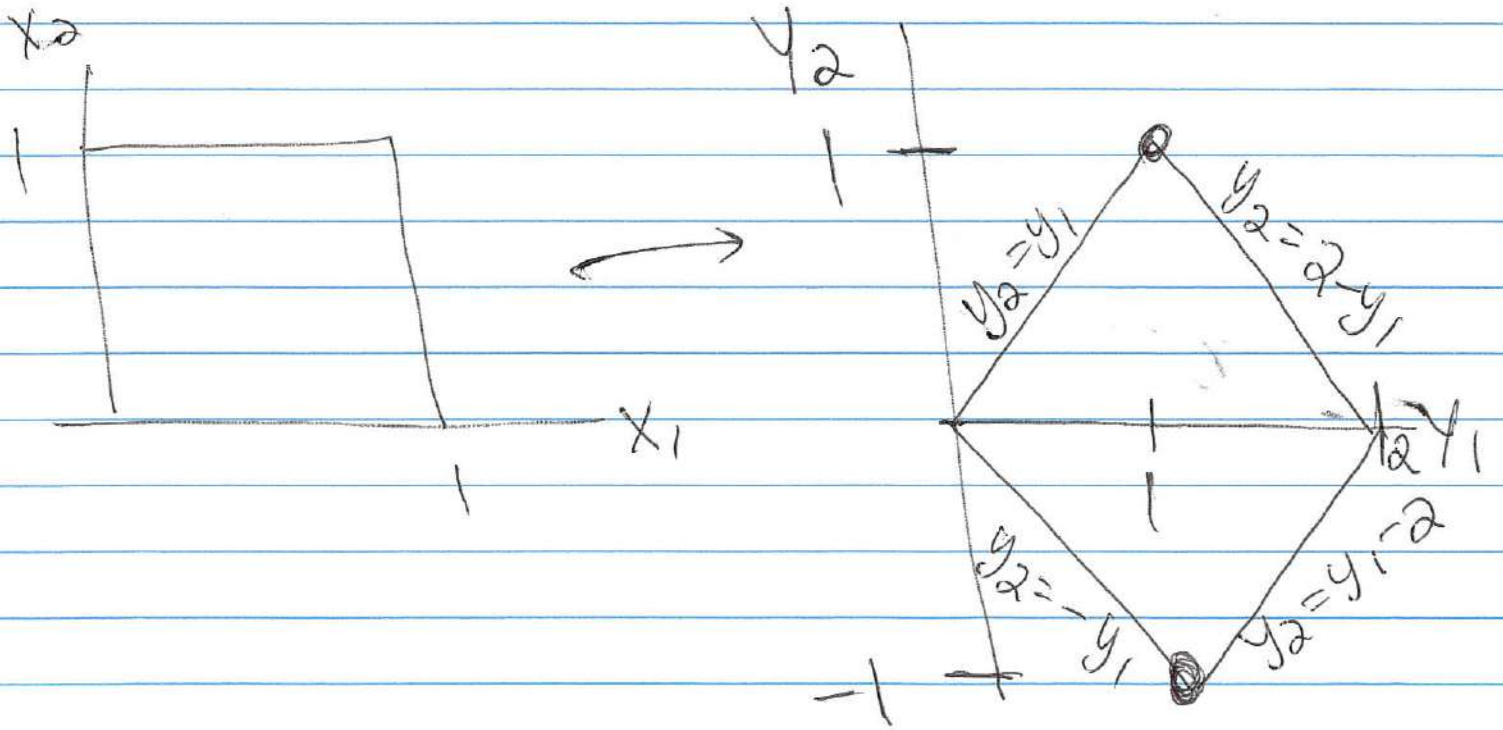
Method 1:

If $x_1=0 \Rightarrow 0=\frac{1}{2}(y_1+y_2) \Rightarrow 0=y_1+y_2$
$\Rightarrow y_2=-y_1$

If $x_1=1 \Rightarrow 1=\frac{1}{2}(y_1+y_2) \Rightarrow 2=y_1+y_2$
$\Rightarrow y_2=2-y_1$

If $x_2=0 \Rightarrow 0=\frac{1}{2}(y_1-y_2) \Rightarrow y_2=y_1$

If $x_2=1 \Rightarrow 1=\frac{1}{2}(y_1-y_2) \Rightarrow y_2=y_1-2$



Method 2:

$0 < x_1 < 1 \Rightarrow 0 < \frac{1}{2}(y_1 + y_2) < 1$

$0 < y_1 + y_2 < 2$

$\boxed{y_2 < 2 - y_1}$

$\boxed{-y_1 < y_2}$

$0 < x_2 < 1 \Rightarrow 0 < \frac{1}{2}(y_1 - y_2) < 1$

$0 < y_1 - y_2 < 2$

$0 > y_2 - y_1 > -2$

$\boxed{y_1 > y_2}$

$\boxed{y_2 > y_1 - 2}$

Marginal Dist for Y_1

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{1,2}(y_1, y_2)\, dy_2$$

$$= \begin{cases} \int_{-y_1}^{y_1} \frac{1}{2}\, dy_2 = y_1, & 0 < y_1 \le 1 \\ \int_{y_1-2}^{2-y_1} \frac{1}{2}\, dy_2 = 2 - y_1, & 1 < y_1 \le 2 \\ 0, & \text{o.w.} \end{cases}$$

Marg. Dist of Y_2

$$f_{Y_2}(y_2) = \int_{-\infty}^{\infty} f_{1,2}(y_1, y_2)\, dy_1$$

$$= \begin{cases} \int_{y_2}^{2-y_2} \frac{1}{2}\, dy_1 = 1 - y_2 & 0 \le y_2 < 1 \\ \int_{-y_2}^{2+y_2} \frac{1}{2}\, dy_1 = y_2 + 1, & -1 < y_2 < 0 \\ 0, & \text{o.w.} \end{cases}$$