Stat 401 - 10/13/17 HW: sec 2.1: 1, 10, 12
sec 2.2: 2, 3, 6
sec 2.3: 2 ← omit #1.

Sec 2.2 - Continued

Moment Generating Function Technique

- if find MGF for $Y = g(X_1, X_2)$ then you know the distribution.

$$\mathbb{E}[Y] = M_Y'(0)$$

$$Var[Y] = M_Y''(0) - [M_Y'(0)]^2$$

ex)

$$p_{1,2}(x_1, x_2) = \begin{cases} \dfrac{\mu_1^{x_1} \mu_2^{x_2} e^{-\mu_1} e^{-\mu_2}}{x_1! \, x_2!}, & x_1 = 0, 1, 2, \ldots \\ & x_2 = 0, 1, 2, \ldots \\ 0, & \text{o.w.} \end{cases}$$

$\mu_1, \mu_2 > 0$ and fixed.

Let $Y = X_1 + X_2$. Find $M_Y(t)$.

$$M_Y(t) = M_{X_1+X_2}(t) = \mathbb{E}\left[e^{t(X_1+X_2)}\right]$$

$$= \sum_{x_1=0}^{\infty} \sum_{x_2=0}^{\infty} \underbrace{e^{t(x_1+x_2)}}_{e^{tx_1} \cdot e^{tx_2}} \cdot \frac{\mu_1^{x_1} \mu_2^{x_2} e^{-\mu_1} e^{-\mu_2}}{x_1! \, x_2!}$$

$$= \left[\sum_{x_1=0}^{\infty} \frac{e^{tx_1}\mu_1^{x_1} e^{-\mu_1}}{x_1!}\right]\left[\sum_{x_2=0}^{\infty} \frac{e^{tx_2}\mu_2^{x_2} e^{-\mu_2}}{x_2!}\right]$$

$$= \left[e^{-\mu_1}\sum_{x_1=0}^{\infty} \frac{(e^t\mu_1)^{x_1}}{x_1!}\right]\left[e^{-\mu_2}\sum_{x_2=0}^{\infty} \frac{(e^t\mu_2)^{x_2}}{x_2!}\right]$$

power series of e^a

$$\sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a$$

$$= \left[e^{-\mu_1} e^{e^t\mu_1}\right]\left[e^{-\mu_2} e^{e^t\mu_2}\right]$$

$$\text{MGF of } X_1 \rightarrow \left[e^{\mu_1(e^t-1)}\right]\left[e^{\mu_2(e^t-1)}\right] \leftarrow \text{MGF of } X_2$$

$$= e^{(e^t-1)(\mu_1+\mu_2)}, \quad -\infty < t < \infty$$

Poisson Dist.

If we know $p_Z(z) = \frac{\mu^z e^{-\mu}}{z!}$, $z = 0, 1, 2, \ldots$

then $M_Z(t) = e^{\mu(e^t-1)}$, $-\infty < t < \infty$

(3)

$$p_Y(y) = \frac{(\mu_1+\mu_2)^y e^{-(\mu_1+\mu_2)}}{y!}$$

$y = 0, 1, 2, \ldots$

marg. dist for X_1 = Poisson.
" " " X_2 = Poisson.

$$\sum_{x_1=0}^{\infty} \frac{(e^t \mu_1)^{x_1}}{x_1!} \qquad \text{let } a = e^t \mu_1$$

$$= \sum_{x_1=0}^{\infty} \frac{a^{x_1}}{x_1!} = \boxed{e^a} = e^{e^t \mu_1}$$

Sec 2.3 – Conditional Dist and Expectations

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} \quad \text{(2 events)}$$

RV

def) Conditional Dist of X_2 given $\underbrace{X_1 = x_1}_{\text{Fixed value}}$

joint pmf

Discrete (X_1, X_2)

$$p_{X_2|X_1}(x_2|x_1) = \frac{p_{X_1,X_2}(x_1, x_2)}{p_{X_1}(x_1)}$$

$p_{X_1}(x_1) > 0$, $x_1 \in D_{X_1}$

$$= \mathbb{P}[X_2 \mid X_1 = x_1]$$

$$= p_{2|1}(x_2|x_1)$$

marg. dist of X_1

Continuous (X_1, X_2)

$$f_{2|1}(x_2|x_1) = \frac{f_{1,2}(x_1, x_2)}{f_1(x_1)}$$

marginal pdf of X_1 $\longleftrightarrow$ $f_1(x_1)$

joint pdf.

$f_1(x_1) > 0$, $x_1 \in D_{X_1}$

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def) Conditional Dist of X_1 given $X_2 = x_2$

Discrete $$p_{1|2}(x_1|x_2) = \frac{p_{1,2}(x_1, x_2)}{p_2(x_2)}$$

where $p_2(x_2) > 0$ and $x_2 \in D_{X_2}$

Continuous $f_{1|2}(x_1|x_2) = \dfrac{f_{1,2}(x_1,x_2)}{f_2(x_2)}$

where $f_2(x_2) > 0$ and $x_2 \in D_{X_2}$

Note) Conditional Prob does satisfy conditions for pmf / pdf.

ex) $p_{1|2}(x_1|x_2) = \dfrac{p_{1,2}(x_1,x_2)}{p_2(x_2)}$ $\leftarrow \geq 0$, $\leftarrow > 0$

$0 \leq p_{1|2}(x_1|x_2) \leq 1$

why?

$\displaystyle\sum_{x_1} p_{1|2}(x_1|\underset{\#}{x_2}) = \sum_{x_1} \frac{p_{1,2}(x_1,x_2)}{p_2(x_2)}$

$\displaystyle = \frac{1}{p_2(x_2)} \overbrace{\sum_{x_1} p_{1,2}(x_1,x_2)}^{\text{Def marg. dist } X_2}$

$\displaystyle = \frac{1}{p_2(x_2)} \cdot p_2(x_2)$

$= 1$

Probabilities:

$$\mathbb{P}\left[a < X_2 < b \mid X_1 = x_1\right] = \mathbb{P}\left[a < X_2 < b \mid x_1\right]$$

$$= \begin{cases} \displaystyle\sum_{a < x_2 < b} p_{2|1}(x_2 \mid x_1) & \text{discrete} \\ \displaystyle\int_a^b f_{2|1}(x_2 \mid x_1)\, dx_2 & \text{continuous} \end{cases}$$

Conditional Expectations:

Let u be a function. Then

$$\mathbb{E}\left[u(X_2) \mid x_1\right] = \begin{cases} \displaystyle\sum_{x_2 \in \mathcal{D}_{X_2}} u(x_2) \cdot p_{2|1}(x_2 \mid x_1) & \text{Discrete} \\ \displaystyle\int_{-\infty}^{\infty} u(x_2) \cdot f_{2|1}(x_2 \mid x_1)\, dx_2 & \text{Continuous} \end{cases}$$

$$\operatorname{Var}\left[X_2 \mid x_1\right] = \mathbb{E}\left[\left\{X_2 - \mathbb{E}\left[X_2 \mid x_1\right]\right\}^2 \,\middle|\, x_1\right]$$

$$= \mathbb{E}\left[X_2^2 \mid x_1\right] - \left(\mathbb{E}\left[X_2 \mid x_1\right]\right)^2$$

ex) Discrete.

2 Bowls:

Bowl 1 → 3 red, 7 blue chips
Bowl 2 → 8 red, 2 blue chips

Flip a coin to decide which bowl to draw a chip from

$$\mathbb{P}[H] = 1/3 \qquad \mathbb{P}[T] = 2/3$$

If Head, draw 1 chip from Bowl 1
If Tails, draw 1 chip from Bowl 2.

a) Define RVs X_1, X_2

b) Find $\mathcal{D}_{X_1}, \mathcal{D}_{X_2}$

c) Find joint dist of (X_1, X_2)

d) Find marg. dist.

e) Find conditional dist of X_1 given $X_2 = \text{blue}$

f) Find $\mathbb{P}[X_1 = \text{tails} \mid X_2 = \text{blue}]$

Sol)

a) Let X_1 = coin flip result
X_2 = color chip drawn.

b) $X_1 = \begin{cases} 1 & \text{if head} \\ -1 & \text{if tails} \end{cases}$

$X_2 = \begin{cases} 0 & \text{if red} \\ 1 & \text{if blue} \end{cases}$

c)

		red D_{X_2} 0	blue 1	$P_{X_1}(x_1)$
D_{X_1}	head 1	$\frac{1}{3}\cdot\frac{3}{10}=\frac{1}{10}=\frac{3}{30}$	$\frac{1}{3}\cdot\frac{7}{10}=\frac{7}{30}$	$\frac{10}{30}$
	tail −1	$\frac{2}{3}\cdot\frac{8}{10}=\frac{16}{30}$	$\frac{2}{3}\cdot\frac{2}{10}=\frac{4}{30}$	$\frac{20}{30}$
$P_{X_2}(x_2)$		$\frac{19}{30}$	$\frac{11}{30}$	1

marg dist for X_1

marg dist X_2.

e) Find $P_{1|2}(X_1|x_2)$ where x_2 = blue.

$$P_{1|2}(x_1|x_2) = \frac{P_{1,2}(x_1,x_2)}{P_2(x_2)} \overset{\text{this ex.}}{=} \frac{P_{1,2}(x_1,1)}{P_2(1)}$$

$$= \frac{p_{1,2}(x_1, 1)}{\frac{11}{30}} = \frac{30}{11}\, p_{1,2}(x_1, 1)$$

for $x_1 = -1, 1$

x_1	-1	1
$p_{1\|2}(x_1\|1)$	$\frac{30}{11} \cdot \frac{4}{30} = \frac{4}{11}$	$\frac{30}{11} \cdot \frac{7}{30} = \frac{7}{11}$

$p_{1,2}(-1,1)$ $\qquad$ $p_{1,2}(1,1)$

f) $\mathbb{P}[\, X_1 = \text{tails} \mid X_2 = \text{blue} \,]$

$= \mathbb{P}[\, X_1 = -1 \mid X_2 = 1 \,]$

$= \frac{4}{11}$