stat 401 - 10/16/17 ①

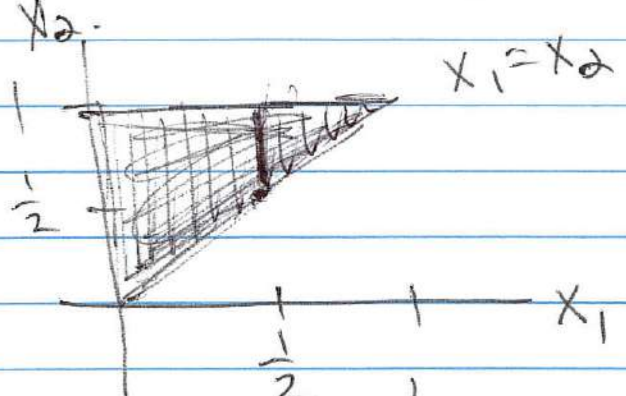
HW: sec 2.3 # 1,4,6   sec 2.5 # 2,3,8
sec 2.4 # 3,10

Sec 2.3 continued

ex) Let X_1, X_2 have the joint pdf

$$f(x_1,x_2) = \begin{cases} 2, & 0 < x_1 < x_2 < 1 \\ 0, & \text{o.w.} \end{cases}$$

a) Find marginal dist. of X_1 and X_2.



Marg. Dist of X_1:

$$\int_{x_1}^{1} 2\,dx_2 = 2x_2\Big|_{x_1}^{1} = 2(1-x_1)$$

$$f_{X_1}(x_1) = \begin{cases} 2(1-x_1), & 0 < x_1 < 1 \\ 0, & \text{o.w.} \end{cases}$$

Marg. Dist of X_2:

$$\int_0^{x_2} 2\,dx_1 = 2x_1\Big|_0^{x_2} = 2x_2$$

$$f_{X_2}(x_2) = \begin{cases} 2x_2, & 0 < x_2 < 1 \\ 0, & \text{o.w.} \end{cases}$$

b) Find conditional dist of X_1 given $X_2 = x_2$, $0 < x_2 < 1$

$$f_{1|2}(x_1|x_2) = \frac{f_{1,2}(x_1, x_2)}{f_2(x_2)} = \frac{2}{2x_2}$$

$\frac{1}{3/4} = \frac{4}{3}$

$$= \begin{cases} \frac{1}{x_2}, & 0 < x_1 < x_2 < 1 \\ 0, & \text{o.w.} \end{cases}$$

c) Find expected value of X_1 given $X_2 = x_2$.

$$E[X_1 | x_2] = \int_0^{x_2} x_1 \cdot \frac{1}{x_2}\, dx_1 = \frac{x_1^2}{2x_2}\Bigg|_0^{x_2}$$

$\#$

$$= \boxed{\frac{x_2}{2}, \quad 0 < x_2 < 1}$$

d) $\mathbb{E}[X_1^2 \mid x_2] = \int_0^{x_2} x_1^2 \cdot \frac{1}{x_2} \, dx_1$

$= \frac{x_1^3}{3x_2} \Big|_0^{x_2} = \boxed{\frac{x_2^2}{3}, \quad 0 < x_2 < 1}$

$\text{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$\text{Var}[X_1 \mid x_2] = \mathbb{E}[X_1^2 \mid x_2] - (\mathbb{E}[X_1 \mid x_2])^2$

$= \frac{x_2^2}{3} - \left(\frac{x_2}{2}\right)^2 = \boxed{\frac{x_2^2}{12}, \quad 0 < x_2 < 1}$

e) $\mathbb{P}[0 < X_1 < \frac{1}{2} \mid X_2 = \frac{3}{4}]$

IS $0 < X_1 < X_2 < 1$

$= \int f_{1|2}(x_1 \mid \frac{3}{4}) \, dx_1$

$= \int_0^{1/2} \frac{4}{3} \, dx_1 = \frac{4x_1}{3} \Big|_0^{1/2} = \boxed{\frac{2}{3}}$

$$\mathbb{P}\left[0 < X_1 < \tfrac{1}{2} \,\middle|\, X_2 = \tfrac{1}{4}\right]$$

$$= \int_0^{1/4} 4\,dx_1 + \int_{1/4}^{1/2} 0\,dx_1$$

$$\tfrac{1}{2} \not< \tfrac{1}{4}$$

f) $\mathbb{P}\left[0 < X_1 < \tfrac{1}{2}\right]$

$$= \int_0^{1/2} 2(1-x_1)\,dx_1 = \frac{3}{4}$$

Note:

- Since $\mathbb{E}[X_2 | x_1]$ is a function of x_1, then $\mathbb{E}[X_2 | \cancel{X_1}]$ is a RV.
  - → possible to find its CDF/PDF or PMF.

ex) Let X_1, X_2 have the joint pdf

$$f_{1,2}(x_1, x_2) = \begin{cases} 6x_2, & 0 < x_2 < x_1 < 1 \\ 0, & \text{o.w.} \end{cases}$$

eventually find $E[X_2 | x_1]$

marg. dist of X_1

$$\int_0^{x_1} 6x_2 \, dx_2 = \frac{6x_2^2}{2} \Big|_0^{x_1} = 3x_1^2$$

$$f_1(x_1) = \begin{cases} 3x_1^2, & 0 < x_1 < 1 \\ 0, & \text{o.w.} \end{cases}$$

cond. dist of X_2 given $X_1 = x_1$

$$f_{2|1}(x_2 | x_1) = \frac{f_{1,2}(x_1, x_2)}{f_1(x_1)} = \frac{6x_2}{3x_1^2}$$

$$= \begin{cases} \dfrac{2x_2}{x_1^2}, & 0 < x_2 < x_1 < 1 \\ 0, & \text{o.w.} \end{cases}$$

$$\mathbb{E}[X_2 \mid x_1] = \int_0^{x_1} x_2 \cdot \frac{2x_2}{x_1^2}\, dx_2 = \int_0^{x_1} \frac{2x_2^2}{x_1^2}\, dx_2$$

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$$= \left.\frac{2x_2^3}{3x_1^2}\right|_0^{x_1} = \frac{2}{3}x_1, \quad 0 < x_1 < 1$$

We know $Y = \mathbb{E}[X_2 \mid X_1] = \frac{2}{3}X_1,$

$0 < x_1 < 1$

Find CDF, PDF, $\mathbb{E}[Y]$, $\mathrm{Var}[Y]$

CDF:

$$\mathbb{P}[Y \le y] = \mathbb{P}\left[\tfrac{2}{3}X_1 \le y\right]$$

$$= \mathbb{P}\left[X_1 \le \tfrac{3}{2}y\right]$$

$$= \int_0^{\frac{3}{2}y} 3x_1^2\, dx_1$$

$$= \left.\frac{3x_1^3}{3}\right|_0^{\frac{3}{2}y} = \frac{27y^3}{8}$$

$$F_Y(y) = \begin{cases} 0, & y < \boxed{0} \\ \frac{27y^3}{8}, & \boxed{0} \le y < \boxed{2/3} \\ 1, & \boxed{2/3} \le y \end{cases}$$

$0 < x_1 < 1$ $\quad Y = \frac{2}{3}X_1$

$\frac{3}{2}Y = X_1$

$0 < \frac{3}{2}Y < 1$

$0 < Y < \frac{2}{3}$

PDF:

$$f_Y(y) = \begin{cases} \frac{81y^2}{8} & 0 < y < \frac{2}{3} \\ 0, & \text{o.w.} \end{cases}$$

$$\mathbb{E}[Y] = \int_0^{2/3} y \cdot \frac{81y^2}{8}\,dy = \frac{1}{2}$$

$$= \mathbb{E}\left[\mathbb{E}[X_2 | X_1]\right] = \mathbb{E}[X_2]$$

$$\mathbb{E}[Y^2] = \int_0^{2/3} y^2 \cdot \frac{81y^2}{8}\,dy = \frac{4}{15}$$

$$\mathrm{Var}[Y] = \frac{4}{15} - \left(\frac{1}{2}\right)^2 = \frac{1}{60}$$

$$= \mathrm{Var}\left[\mathbb{E}[X_2 | X_1]\right]$$

Thm) Let (X_1, X_2) be a random vector such that the variance of X_2 is finite. Then

a) $\mathbb{E}\left[\mathbb{E}[X_2 | X_1]\right] = \mathbb{E}[X_2]$

b) $Var\left[\mathbb{E}[X_2 | X_1]\right] \le Var[X_2]$

ex) Marg. Dist of X_2

$$\int_{x_2}^{1} 6x_2 \, dx_1 = 6x_2(1-x_2)$$

$$f_2(x_2) = \begin{cases} 6x_2(1-x_2), & 0 < x_2 < 1 \\ 0, & \text{o.w.} \end{cases}$$

$$\mathbb{E}[X_2] = \int_0^1 x_2 \cdot 6x_2(1-x_2) \, dx_2$$

$$= \frac{1}{2}$$

$Var[X_2] = \frac{3}{60}$. $\quad Var[Y] = \frac{1}{60} \le \frac{3}{60} = Var[X_2]$