Stat 401 - 10/20/17

Sec 2.5 - Independent Random Variables

For continuous case

$$f_{2|1}(x_2|x_1) = \frac{f_{1,2}(x_1, x_2)}{f_1(x_1)}$$

$$\Rightarrow \quad f_{1,2}(x_1, x_2) = f_{2|1}(x_2|x_1) \cdot f_1(x_1)$$

Suppose $f_{2|1}(x_2|x_1)$ does not depend on x_1.

Then

$$f_{2|1}(x_2|x_1) = f_2(x_2)$$

Proof

$$f_{2|1}(x_2|x_1) = f_{2|1}(x_2|x_1) \underbrace{\int_{-\infty}^{\infty} f_1(x_1)\,dx_1}_{=1}$$

doesn't depend on x_1

$$= \int_{-\infty}^{\infty} \underbrace{f_{2|1}(x_2|x_1) \cdot f_1(x_1)}\,dx_1$$

$$= \int_{-\infty}^{\infty} f_{1,2}(x_1, x_2)\,dx_1$$

$$= f_2(x_2)$$

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$\therefore f_{2|1}(x_2|x_1) = f_2(x_2)$

and $f(x_1, x_2) = f_1(x_1) \cdot f_2(x_2)$

def) Independence

Let X_1, X_2 have joint pdf
$f_{1,2}(x_1, x_2)$ [joint pmf $p_{1,2}(x_1,x_2)$]

and the marginal pdfs [pmfs]
$f_1(x_1)$ [$p_1(x_1)$] and $f_2(x_2)$ [$p_2(x_2)$]

then random variables X_1 and X_2 are said to be independent

if and only if

$$f_{1,2}(x_1, x_2) = f_1(x_1) f_2(x_2)$$

$$p_{1,2}(x_1, x_2) = p_1(x_1) p_2(x_2).$$

If not independent, then they are dependent.

ex) Assume X_1, X_2 are independent with

$$f_1(x_1) = \begin{cases} \frac{1}{2} x_1, & 0 < x_1 < 1 \\ 0, & \text{o.w.} \end{cases}$$

$$f_2(x_2) = \begin{cases} \frac{1}{3} x_2^2, & 0 < x_2 < 1 \\ 0, & \text{o.w.} \end{cases}$$

Then

$$f_{1,2}(x_1, x_2) = \begin{cases} \frac{1}{6} x_1 x_2^2, & \begin{matrix} 0 < x_1 < 1 \\ 0 < x_2 < 1 \end{matrix} \\ 0, & \text{o.w.} \end{cases}$$

<u>Thm)</u> X_1 and X_2 are independent iff $f(x_1, x_2)$ can be written as a product of a nonnegative function of x_1 and a nonnegative function of x_2.

$$f(x_1, x_2) \equiv g(x_1) h(x_2)$$

where $g(x_1) > 0$, $x_1 \in S_1$

and $h(x_2) > 0$, $x_2 \in S_2$

ex) $f(x_1, x_2) = \begin{cases} x_1 + x_2, & 0 < x_1 < 1, \; 0 < x_2 < 1 \\ 0, & \text{o.w.} \end{cases}$

Are X_1, X_2 independent?

$$f(x_1) = \begin{cases} \int_0^1 (x_1 + x_2)\,dx_2 = x_1 + \frac{1}{2}, & 0 < x_1 < 1 \\ 0, & \text{o.w.} \end{cases}$$

$$f(x_2) = \begin{cases} \int_0^1 (x_1 + x_2)\,dx_1 = x_2 + \frac{1}{2}, & 0 < x_2 < 1 \\ 0, & \text{o.w.} \end{cases}$$

Is $f(x_1, x_2) = f_1(x_1) \cdot f_2(x_2)$?

$$x_1 + x_2 \neq (x_1 + \tfrac{1}{2}) \cdot (x_2 + \tfrac{1}{2})$$

NOT Independent.

They are dependent.

Thm: Let (X_1, X_2) have joint cdf $F(x_1, x_2)$ and let X_1 and X_2 have marginal CDFs $F(x_1)$ and $F(X_2)$

X_1 and X_2 are independent

iff $F(x_1, x_2) = F_1(x_1)\, F_2(x_2)$
$\forall (x_1, x_2) \in \mathbb{R}^2.$

Proof: $(\Leftarrow)$ Suppose $F(x_1, x_2) = F(x_1)F(x_2)$.

Then

$$f(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F(x_1, x_2)$$

$$\overset{\text{I.A.}}{=} \frac{\partial^2}{\partial x_1 \partial x_2} F(x_1) F(x_2)$$

$$= \left[\frac{\partial}{\partial x_1} F(x_1)\right]\left[\frac{\partial}{\partial x_2} F(x_2)\right]$$

$$= f(x_1) \cdot f(x_2)$$

$\therefore$ X_1 and X_2 are independent.

($\Rightarrow$) Assume X_1, X_2 independent

$$F(x_1, x_2) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} f(w_1, w_2)\, dw_2\, dw_1$$

$$\overset{\text{I.A.}}{=} \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} f(w_1) f(w_2)\, dw_2\, dw_1$$

$$= \left[\int_{-\infty}^{x_1} f(w_1)\, dw_1\right]\left[\int_{-\infty}^{x_2} f(w_2)\, dw_2\right]$$

$$= F(x_1)\, F(x_2)$$

$\square$

Thm) X_1 and X_2 are independent iff

$$\mathbb{P}[a < X_1 \leq b, c < X_2 \leq d]$$
$$= \mathbb{P}[a < X_1 \leq b] \cdot \mathbb{P}[c < X_2 \leq d]$$

for every $\underbrace{a<b,\ c<d}_{\text{all constants.}}$

Thm) If X_1 and X_2 are independent then $\mathbb{E}[u(X_1)v(X_2)] = \mathbb{E}[u(X_1)] \cdot \mathbb{E}[v(X_2)]$

Proof)

Assume X_1, X_2 are independent.

$$\mathbb{E}[u(X_1)v(X_2)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(X_1)v(X_2) \cdot f_{1,2}(x_1,x_2)\,dx_1\,dx_2$$

indep.

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(X_1)v(X_2)\, f_1(x_1) f_2(x_2)\,dx_1\,dx_2$$

$$= \left[\int_{-\infty}^{\infty} u(X_1) f_1(x_1)\,dx_1\right]\left[\int_{-\infty}^{\infty} v(X_2) f_2(X_2)\,dx_2\right]$$

$$= \mathbb{E}[u(X_1)] \cdot \mathbb{E}[v(X_2)]$$

Note) If X, Y are independent then
$$\mathrm{Cov}(X, Y) = 0$$

Proof)
$$\mathrm{Cov}(X,Y) = \mathbb{E}\left[(X-\mu_x)(Y-\mu_y)\right]$$
$$\overset{\text{indep}}{=} \mathbb{E}[X-\mu_x]\,\mathbb{E}[Y-\mu_y]$$
$$= \Big\{\mathbb{E}[X] - \underbrace{\mathbb{E}[X]}_{\mu_x}\Big\}\Big\{\mathbb{E}[Y] - \underbrace{\mathbb{E}[Y]}_{\mu_y}\Big\}$$
$$= 0$$

Converse not generally true.

ex) Let (X, Y) have joint pmf

$$p_{X,Y}(x,y) = \begin{cases} \frac{1}{8}, & x = -1, 0, 1 \\ & y = -1, 0, 1 \\ & \text{but } (x,y) \neq (0,0) \\ 0., & \text{o.w.} \end{cases}$$

		$y=-1$	$y=0$	$y=1$	$p(x)$
X	-1	$1/8$	$1/8$	$1/8$	$3/8$
	0	$1/8$	0	$1/8$	$2/8$
	1	$1/8$	$1/8$	$1/8$	$3/8$
$p(y)$		$\frac{3}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	1

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0$$

$$\mathbb{E}[XY] = \sum_x \sum_y xy\, p(x,y) = 0$$

$$= -1(-1)\left(\tfrac{1}{8}\right) + (-1)(1)\left(\tfrac{1}{8}\right) + 0 + 0 + 0 + 1(-1)\left(\tfrac{1}{8}\right) + 0 + 1(1)\left(\tfrac{1}{8}\right)$$

$$\mathbb{E}[X] = -1\left(\tfrac{3}{8}\right) + 0\left(\tfrac{2}{8}\right) + 1\left(\tfrac{3}{8}\right) = 0$$

$$= \mathbb{E}[Y]$$

Indep ? $\underbrace{p(x,y)}_{\frac{1}{8}} = p(x)p(y)$

$$\underset{\frac{1}{8}}{\mathbb{P}[X=1, Y=1]} \overset{?}{=} \underset{\frac{3}{8}}{\mathbb{P}[X=1]} \cdot \underset{\frac{3}{8}}{\mathbb{P}[Y=1]}$$

$$\frac{1}{8} \neq \frac{3}{8} \cdot \frac{3}{8}$$

Dependent

Thm) Suppose the joint MGF $M(t_1, t_2)$ exists for RVs X_1, X_2.

Then X_1 and X_2 are independent iff

$$M(t_1, t_2) = M(t_1, 0)\, M(0, t_2)$$

i.e. you can write the joint MGF as a product of the marginal MGFs.

ex) in sec 2.2

$$p_{1,2}(x_1, x_2) = \begin{cases} \dfrac{\mu_1^{x_1} \mu_2^{x_2} e^{-\mu_1} e^{-\mu_2}}{x_1!\, x_2!}, & x_1 = 0,1,2,\ldots \quad x_2 = 0,1,2,\ldots \\ 0, & \text{o.w.} \end{cases}$$

$$M(t_1, t_2) = \mathbb{E}\left[e^{t_1 X_1 + t_2 X_2}\right] = \sum_{x_1=0}^{\infty} \sum_{x_2=0}^{\infty} e^{t_1 x_1 + t_2 x_2} \frac{\mu_1^{x_1} \mu_2^{x_2} e^{-\mu_1} e^{-\mu_2}}{x_1!\, x_2!}$$

$$= e^{\mu_1(e^{t_1} - 1)}\, e^{\mu_2(e^{t_2} - 1)}$$

$$M(t_1,0) = \mathbb{E}\left[e^{t_1 X_1}\right]$$

$$= \sum_{x_1=0}^{\infty}\sum_{x_2=0}^{\infty} e^{t_1 x_1} \frac{\mu_1^{x_1}\mu_2^{x_2} e^{-\mu_1} e^{-\mu_2}}{x_1!\, x_2!}$$

$$= \sum_{x_1=0}^{\infty} \frac{e^{t_1 x_1} e^{-\mu_1}\mu_1^{x_1}}{x_1!} \underbrace{\sum_{x_2=0}^{\infty} \frac{\mu_2^{x_2} e^{-\mu_2}}{x_2!}}_{=1}$$

$$= e^{-\mu_1} \sum_{x_1=0}^{\infty} \frac{e^{t_1 x_1}\mu_1^{x_1}}{x_1!}$$

$$= e^{-\mu_1} \sum_{x_1=0}^{\infty} \frac{(e^{t_1}\mu_1)^{x_1}}{x_1!}$$

$$= e^{-\mu_1} e^{\mu_1 e^{t_1}} = e^{\mu_1(e^{t_1}-1)}$$

since $\sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a$

Similarly $M(0,t_2) = e^{\mu_2(e^{t_2}-1)}$

$\therefore M(t_1,t_2) = M(t_1,0)\,M(0,t_2)$ and X_1, X_2 are independent.