Stat 401 - 10/23/17 Sec 2.7 - 1, 4 ①

HW Sec 2.6: 1a, c, d, f, g ; 3 Sec 2.7 - 8b (use MGF doc. to help)

Sec 2.6 - Extension to Several RVs

def) Suppose we have a random experiment with sample space $\mathcal{C}$

Suppose $X_i(c) = x_i$, for $i = 1, 2, \ldots, n$

$(X_1, X_2, \ldots, X_n) \rightarrow$ is an n-dimensional random vector

The space of random vector is

$$\mathcal{D} = \{(x_1, x_2, \ldots, x_n) : x_1 = X_1(c), \ldots, x_n = X_n(c), \text{ for } c \in \mathcal{C}\}$$

def) The column vector $\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}_{n \times 1}$

List of RVs

the column vector $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$

observed values

<u>def</u> Joint CDF

$$F_{\vec{X}}(\vec{x}) = \mathbb{P}[X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n]$$

$$= \begin{cases} \displaystyle\sum_{\omega_1 \le x_1} \cdots \sum_{\omega_n \le x_n} p(\omega_1, \ldots, \omega_n) & \text{if discrete} \\ \displaystyle\int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_n} f(\omega_1, \ldots, \omega_n)\, d\omega_n \cdots d\omega_1 & \text{if continuous} \end{cases}$$

For continuous case.

$$\frac{\partial^n}{\partial x_1 \cdots \partial x_n} F_{\vec{X}}(\vec{x}) = f_{\vec{X}}(\vec{x})$$

Note: $\displaystyle\sum_{x_1} \cdots \sum_{x_n} \overbrace{p(x_1, \ldots, x_n)}^{\ge 0} = 1$

$$\int_{x_n} \cdots \int_{x_1} \underbrace{f(x_1, \ldots, x_n)}_{\ge 0}\, dx_1 \cdots dx_n = 1$$

ex) Let

$$f(x,y,z) = \begin{cases} e^{-(x+y+z)}, & 0<x,y,z<\infty \\ 0, & \text{o.w.} \end{cases}$$

joint pdf of X, Y, Z

Find the CDF.

$$F(x,y,z) = \mathbb{P}[X \le x, Y \le y, Z \le z]$$

$$= \int_0^x \int_0^y \int_0^z \underbrace{e^{-(u+v+w)}}_{e^{-u}e^{-v}e^{-w}} \, dw\,dv\,du$$

$$= \left[\int_0^x e^{-u}\,du\right]\left[\int_0^y e^{-v}\,dv\right]\left[\int_0^z e^{-w}\,dw\right]$$

$$= \begin{cases} (1-e^{-x})(1-e^{-y})(1-e^{-z}), & 0 \le x,y,z < \infty \\ 0, & \text{o.w.} \end{cases}$$

def) Let $(X_1, \ldots, X_n)$ be a random vector.

Let $Y = u(X_1, \ldots, X_n)$
$\uparrow$ function

$$\mathbb{E}[Y] = \begin{cases} \displaystyle\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} u(x_1, \ldots, x_n) \cdot f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n & \text{if continuous} \\ \displaystyle\sum_{x_1} \cdots \sum_{x_n} u(x_1, \ldots, x_n)\, p(x_1, \ldots, x_n) & \text{if discrete} \end{cases}$$

$$\mathbb{E}\left[\sum_{j=1}^{m} k_j Y_j\right] = \sum_{j=1}^{m} k_j \mathbb{E}[Y_j]$$

$\uparrow$ constants

ex) $\mathbb{E}[5X_1 + 2X_2 - 3X_3^2 X_4]$

$= 5\mathbb{E}[X_1] + 2\mathbb{E}[X_2] - 3\mathbb{E}[X_3^2 X_4]$

def) Marginal Dist of X_1

$$f_1(x_1) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n)\, dx_2 \ldots dx_n$$

Marginal Dist of X_2

$$f_2(x_2) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n)\, dx_1\, dx_3 \cdots dx_n$$

Marginal Dist of X_1 and X_2

$$f(x_1, x_2) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n)\, dx_3 \cdots dx_n$$

Marginal Dist of X_1, X_3, X_4

$$f(x_1, x_3, x_4) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n)\, dx_2\, dx_5 \cdots dx_n$$

def) Joint Conditional PDF of $X_2, \ldots, X_n$ given $X_1 = x_1$.

$$f(x_2, \ldots, x_n \mid x_1) = \frac{f(x_1, \ldots, x_n)}{f(x_1)}$$

Joint Conditional PDF of $X_3, \ldots, X_n$ given $X_1 = x_1$ and $X_2 = x_2$

$$f(x_3, \ldots, x_n \mid x_1, x_2) = \frac{f(x_1, \ldots, x_n)}{f(x_1, x_2)}$$

ex)

$$f(x_1, x_2, x_3, x_4) = \begin{cases} e^{-(x_1+x_2+x_3+x_4)} & , \ 0 < x_1, x_2, x_3, x_4 < \infty \\ 0 & , \ \text{o.w.} \end{cases}$$

Find $f(x_3 \mid x_1, x_2, x_4) = \dfrac{f(x_1, x_2, x_3, x_4)}{f(x_1, x_2, x_4)}$

$$f(x_1, x_2, x_4) = \int_0^\infty e^{-(x_1+x_2+x_3+x_4)} \, dx_3$$
$$= e^{-(x_1+x_2+x_4)} \int_0^\infty e^{-x_3} \, dx_3$$

$$= \begin{cases} e^{-(x_1+x_2+x_4)}, & \text{for } 0 < x_1, x_2, x_4 < \infty \\ 0, & \text{o.w.} \end{cases}$$

$$= \frac{e^{-(x_1+x_2+x_3+x_4)}}{e^{-(x_1+x_2+x_4)}} = e^{-x_3} \quad \text{for } 0 < x_3 < \infty$$

def) Conditional Expectation of $u(X_2, \dots, X_n)$ given $X_1 = x_1$

$$\mathbb{E}\left[u(X_2, \dots, X_n) \,\middle|\, x_1\right] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} u(x_2, \dots, x_n) f_{2,\dots,n|1}(x_2, \dots, x_n | x_1) \, dx_2 \cdots dx_n$$

provided $f_1(x_1) > 0$.

ex) $\mathbb{E}\left[X_3 X_4 \,\middle|\, x_1, x_2\right]$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_3 x_4 \cdot f_{3,4|1,2}(x_3, x_4 | x_1, x_2) \, dx_3 \, dx_4$$

Note: $\mathbb{E}\left[u(X_2,\dots,X_n) \,\middle|\, X_1\right]$ is a RV.

see Sec 2.3 for why

$\mathbb{E}\left[X_2 \,\middle|\, X_1\right]$ is a RV.

<u>def)</u> $X_1, X_2, \dots, X_n$ are mutually independent iff

$$f(x_1,\dots,x_n) = f_1(x_1)\cdots f_n(x_n)$$

OR

$$p(x_1,\dots,x_n) = p_1(x_1)\cdots p_n(x_n)$$

mutually indep $\Rightarrow$ pairwise indep

pairwise indep $\not\Rightarrow$ mutually indep.

<u>def)</u> The MGF of $X_1, \dots, X_n$ is given by

$$M(t_1,\dots,t_n) = \mathbb{E}\left[e^{t_1X_1 + t_2X_2 + \cdots + t_nX_n}\right]$$

must be finite

→ it is unique

→ it completely determines the joint dist.

Thm) Suppose $X_1, \ldots, X_n$ are n, mutually indep RVs.

Let MGFs tobe $M_{X_i}(t)$

The MGF for $\sum_{i=1}^{n} k_i X_i$ is

$$\prod_{i=1}^{n} M_{X_i}(k_i t)$$

i.e.

$$M_{k_1X_1 + k_2X_2 + \cdots + k_nX_n}(t)$$

$$= \prod_{i=1}^{n} M_{X_i}(k_i t)$$

ex) $M_{X_1 + \cdots + X_n}(t) = M_{X_1}(t) \cdot \ldots \cdot M_{X_n}(t)$