Stat 401 - 10/25/17

~~[illegible]~~

Sec 2.6 - one last definition

<u>def</u>] iid = RVs that are mutually <u>independent</u> and are <u>identically</u> <u>distributed</u>

Sec 2.7 - Transformations for Several RVs

Continuous

Steps:

Let $f(x_1, \ldots, x_n)$ be the joint pdf of $X_1, \ldots, X_n$.

(*) Define
$$y_1 = u_1(x_1, \ldots, x_n)$$
$$y_2 = u_2(x_1, \ldots, x_n)$$
$$\vdots$$
$$y_n = u_n(x_1, \ldots, x_n)$$

functions

This is a 1-1 transformation.

(*) Solve for x_i's. functions
$$x_1 = w_1(y_1, \ldots, y_n)$$
$$x_2 = w_2(y_1, \ldots, y_n)$$
$$\vdots$$
$$x_n = w_n(y_1, \ldots, y_n)$$

* Find Jacobian = J

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}$$ ← determinant

* Find $|J|$ ← abs value

* Find $D_{\vec{Y}}$, the space for $\vec{Y} = (Y_1, \ldots, Y_n)$

* The joint pdf of $\vec{Y}$ is given as

$$f_{\vec{Y}}(y_1, \ldots, y_n) = \begin{cases} f_{\vec{X}}(x_1, \ldots, x_n)|J| \\ = f_{\vec{X}}\big(w_1(y_1, \ldots, y_n), \ldots, w_n(y_1, \ldots, y_n)\big)|J| & \text{for } (y_1, \ldots, y_n) \in D_{\vec{Y}} \\ 0, & \text{o.w.} \end{cases}$$

ex) Let X_1, X_2, X_3 have joint pdf

$$f(x_1, x_2, x_3) = \begin{cases} 48\, x_1 x_2 x_3, & 0 < x_1 < x_2 < x_3 < 1 \\ 0, & \text{o.w.} \end{cases}$$

Suppose $Y_1 = \dfrac{X_1}{X_2}$, $Y_2 = \dfrac{X_2}{X_3}$, $Y_3 = X_3$

Find the joint pdf of $\vec{Y}$

* Solve for x_i

$x_3 = y_3$

$x_2 = y_2 x_3 = y_2 y_3$

$x_1 = y_1 x_2 = y_1 y_2 y_3$

* Find J.

$$J = \begin{vmatrix} y_2 y_3 & y_1 y_3 & y_1 y_2 \\ 0 & y_3 & y_2 \\ 0 & 0 & 1 \end{vmatrix} = y_2 y_3 \begin{vmatrix} y_3 & y_2 \\ 0 & 1 \end{vmatrix} - 0 + 0$$

$$= y_2 y_3 (y_3 - 0 y_2) = y_2 y_3^2$$

* Find $D_{\vec{y}}$

$$0 < x_1 < x_2 < x_3 < 1$$

$$0 < y_1 y_2 y_3 < y_2 y_3 < y_3 < 1$$

SO

A) $0 < \dfrac{y_1 y_2 y_3}{y_2 y_3} < \dfrac{y_2 y_3}{y_2 y_3}$

$$0 < y_1 < 1$$

B) $0 < y_2 y_3 < y_3$

$$0 < y_2 < 1$$

C) $0 < y_3 < 1$

* Find $|J| = |y_2 y_3^2| = y_2 y_3^2$

⊛ Find $f_{\vec{X}}(x_1, x_2, x_3)\,|J|$

$$= f_{\vec{X}}(y_1y_2y_3, y_2y_3, y_3)\cdot y_2y_3^2$$

$$= 48(y_1y_2y_3)(y_2y_3)(y_3)\cdot y_2y_3^2$$

$$= 48\,y_1\,y_2^3\,y_3^5$$

⊛ Joint pdf of $\vec{Y}$ is

$$f_{\vec{y}}(y_1, y_2, y_3) = \begin{cases} 48y_1y_2^3y_3^5, & 0<y_1,y_2,y_3<1 \\ 0, & \text{o.w.} \end{cases}$$

$\boxed{Q}$ Are Y_1, Y_2, Y_3 mutually independent?

→ check if

$$f(y_1, y_2, y_3) = f_1(y_1)f_2(y_2)f_3(y_3)$$

⊛ Find $f_1(y_1)$

$$\int_0^1 \int_0^1 48\, y_1 y_2^3 y_3^5 \, dy_2 \, dy_3$$

$$= 48 y_1 \left[\int_0^1 y_2^3 \, dy_2\right]\left[\int_0^1 y_3^5 \, dy_3\right]$$

$$= 48 y_1 \left. \frac{y_2^4}{4} \right|_0^1 \left. \frac{y_3^6}{6} \right|_0^1$$

$$= \begin{cases} 2y_1, & 0 < y_1 < 1 \\ 0, & \text{o.w.} \end{cases}$$

⊛ Find $f_2(y_2)$

$$\int_0^1 \int_0^1 48\, y_1 y_2^3 y_3^5 \, dy_1 \, dy_3$$

$$= \begin{cases} 4y_2^3, & 0 < y_2 < 1 \\ 0, & \text{o.w.} \end{cases}$$

(*) Find $f_3(y_3)$

$$\int_0^1\int_0^1 48y_1y_2^3y_3^5\,dy_1\,dy_2$$

$$= \begin{cases} 6y_3^5, & 0<y_3<1 \\ 0, & o.w. \end{cases}$$

(*) $f(y_1)f(y_2)f(y_3)$

$$= 2y_1 \cdot 4y_2^3 \cdot 6y_3^5$$

$$= 48y_1y_2^3y_3^5$$

$$= f(y_1,y_2,y_3)$$

$\therefore$ Y_1, Y_2, Y_3 are mutually independent.

OR. $f(y_1,y_2,y_3) \equiv g(y_1)h(y_2)i(y_3)$

ex) Let X_1, X_2, X_3 be iid with common pdf

$$f(x) = \begin{cases} e^{-x}, & 0 < x < \infty \\ 0, & o.w. \end{cases}$$

Define $Y_1 = \dfrac{X_1}{X_1+X_2+X_3}$, $Y_2 = \dfrac{X_2}{X_1+X_2+X_3}$,

$$Y_3 = X_1 + X_2 + X_3$$

Q Are Y_1, Y_2, Y_3 independent?

* Solve for x_i's.

$$x_1 = y_1(x_1+x_2+x_3) = y_1 y_3$$

$$x_2 = y_2 y_3$$

$$= y_2(x_1+x_2+x_3)$$

$$x_3 = y_3 - x_1 - x_2$$

$$= y_3 - y_1 y_3 - y_2 y_3$$

$$= y_3(1 - y_1 - y_2)$$

(4) Find J

$$J = \begin{vmatrix} y_3 & 0 & y_1 \\ 0 & y_3 & y_2 \\ -y_3 & -y_3 & 1-y_1-y_2 \end{vmatrix}$$

$$= y_3 \begin{vmatrix} y_3 & y_2 \\ -y_3 & 1-y_1-y_2 \end{vmatrix} - 0 \begin{vmatrix} 0 & y_2 \\ -y_3 & 1-y_1-y_2 \end{vmatrix}$$

$$+ y_1 \begin{vmatrix} 0 & y_3 \\ -y_3 & -y_3 \end{vmatrix}$$

$$= y_3\left(y_3(1-y_1-y_2) - -y_2 y_3 \right) - 0$$

$$+ y_1\left(0 - -y_3^2 \right)$$

$$= y_3^2 = |J|$$

(4) Find $D_{\vec{Y}}$.

$0 < x_1 < \infty \qquad 0 < x_2 < \infty \qquad 0 < x_3 < \infty$

(1) $0 < y_1 y_3 < \infty$ (2) $0 < y_2 y_3 < \infty$ (3) $0 < y_3(1-y_1-y_2) < \infty$

	Case 1	Case 2
(1)	$y_1, y_3 > 0$	$y_1, y_3 < 0$ ← $y_1 < 0$
(2)	$y_2, y_3 > 0$	$y_2, y_3 < 0$ ← $y_2 < 0$
(3)	$y_3 > 0$	$y_3 < 0$
	$1 - y_1 - y_2 > 0$	$1 - y_1 - y_2 < 0$

$\Rightarrow 1 - (y_1 + y_2) > 0 \qquad \Rightarrow 1 - (y_1 + y_2) < 0$

$\Rightarrow 1 > y_1 + y_2 > 0 \qquad \Rightarrow 1 < y_1 + y_2$

correct choice

Can't happen

$$D_{\vec{Y}} = \left\{ (y_1, y_2, y_3) : \begin{array}{c} y_1 > 0,\ y_2 > 0,\ y_3 > 0 \\ 0 < y_1 + y_2 < 1 \end{array} \right\}$$

$$f_{\vec{X}}(x_1, x_2, x_3) = e^{-x_1} e^{-x_2} e^{-x_3}$$

$$= e^{-(x_1+x_2+x_3)}$$

* Joint pdf of $\vec{Y}$.

$$f_{\vec{X}}(x_1, x_2, x_3)\,|J|$$

$$= f_{\vec{X}}(y_1y_3,\ y_2y_3,\ y_3 - y_1y_3 - y_2y_3) \cdot y_3^2$$

$$= e^{-(\cancel{y_1y_3} + \cancel{y_2y_3} + y_3 - \cancel{y_1y_3} - \cancel{y_2y_3})} \cdot y_3^2$$

$$= \begin{cases} e^{-y_3}\, y_3^2, & y_1 > 0,\ y_2 > 0,\ y_3 > 0,\quad 0 < y_1 + y_2 < 1 \\ 0, & \text{o.w.} \end{cases}$$