Stat 401 – 10/27/17

Sec 2.7 continued

$$f_{\vec{X}}(x_1, x_2, x_3) \underset{\text{indep.}}{=} f(x_1)f(x_2)f(x_3)$$

$$\underset{\text{distributed}}{\overset{\text{identically}}{\longrightarrow}} = e^{-x_1} e^{-x_2} e^{-x_3}$$

$$= e^{-(x_1 + x_2 + x_3)}$$

for $x_1, x_2, x_3 > 0$

Joint pdf for $\vec{Y}$

$$= \begin{cases} e^{-y_3} y_3^2 , & \begin{matrix} y_1 > 0 \\ y_2 > 0 \\ y_3 > 0 \end{matrix} \quad 0 < y_1 + y_2 < 1 \\ 0 , & \text{o.w.} \end{cases}$$

Independence?

Is $f(y_1, y_2, y_3) = f(y_1)f(y_2)f(y_3)$?

⊛ Find $f(y_1)$

$$\int_0^\infty \int_0^{1-y_1} e^{-y_3} y_3^2 \, dy_2 \, dy_3$$

$$0 < y_1 + y_2 < 1$$
$$-y_1 < y_2 < 1 - y_1$$
$$0 < y_2 < 1 - y_1$$

$$= \int_0^\infty y_2 y_3^2 e^{-y_3} \Big|_0^{1-y_1} dy_3$$

$$= \int_0^\infty (1-y_1) y_3^2 e^{-y_3} \, dy_3$$

$$= (1-y_1) \int_0^\infty y_3^2 e^{-y_3} \, dy_3$$

$$= (1-y_1)\left[-y_3^2 e^{-y_3} - 2y_3 e^{-y_3} - 2e^{-y_3} \Big|_0^\infty\right]$$

$$\begin{array}{ll} y_3^2 & +e^{-y_3} \\ 2y_3 & -e^{-y_3} \\ 2 & +e^{-y_3} \\ 0 & -e^{-y_3} \end{array}$$

$$= (1-y_1)\left[0 - 0 - 0 - \left(0 - 0 - 2\right)\right]$$

$$= (1-y_1)(2)$$

$$f(y_1) = \begin{cases} 2(1-y_1), & 0 < y_1 < 1 \\ 0, & \text{o.w.} \end{cases}$$

$\int_0^1 2(1-y)\,dy$

$2\int_0^1 (1-y)\,dy$

$2 \quad y - \frac{y^2}{2}\Big|_0^1$

$2(1-\frac{1}{2})$

$= 2(\frac{1}{2})$

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⊛ Find $f(y_2)$

$$\int_0^{\infty}\int_0^{1-y_2} e^{-y_3}\, y_3^2 \, dy_1\, dy_3$$

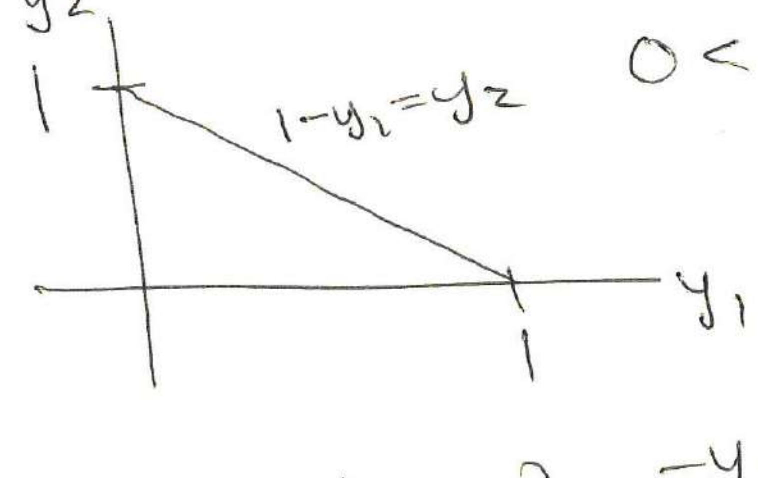
$$f(y_2) = \begin{cases} 2(1-y_2), & 0 < y_2 < 1 \\ 0, & \text{o.w.} \end{cases}$$

⊛ Find $f(y_3)$

$$\int_0^1 \int_0^{1-y_2} e^{-y_3}\, y_3^2 \, dy_1\, dy_2$$

$0 < y_1 + y_2 < 1$

$0 < y_2 < 1$



$$= \frac{1}{2} y_3^2 e^{-y_3}$$

$$f(y_3) = \begin{cases} \frac{1}{2} y_3^2 e^{-y_3}, & 0 < y_3 < \infty \\ 0, & \text{o.w.} \end{cases}$$

$\circledast$ $f(y_1)f(y_2)f(y_3)$

$$= 2(1-y_1)\cdot\cancel{2}(1-y_2)\cdot\cancel{\tfrac{1}{2}}y_3^2e^{-y_3}$$
$$= 2(1-y_1)(1-y_2)y_3^2e^{-y_3}$$
$$\neq y_3^2e^{-y_3}$$
$$= f(y_1,y_2,y_3)$$

NOT independent

i.e. Dependent.

ex) $f(x_1,x_2)=\begin{cases}\frac{1}{\pi}, & 0<x_1^2+x_2^2<1\\ 0, & \text{o.w.}\end{cases}$

Define $Y_1 = X_1^2 + X_2^2$.

$$Y_2 = \frac{X_1^2}{X_1^2+X_2^2}$$

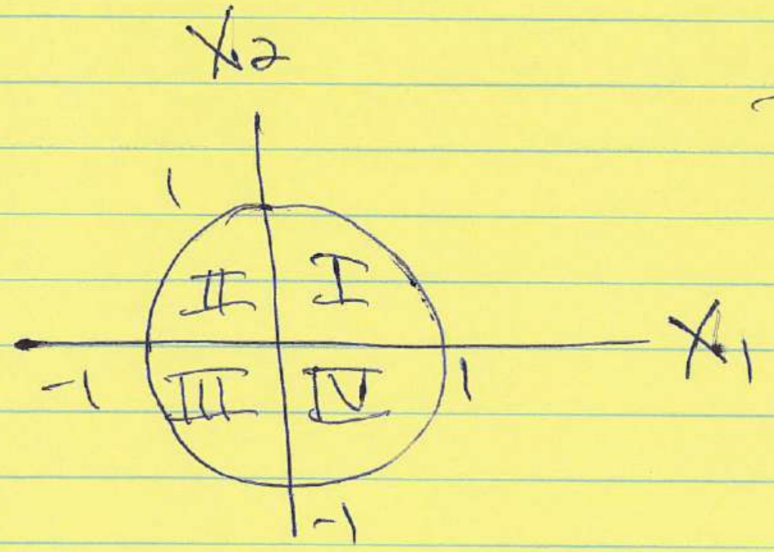
Find $f(y_1,y_2)$

(4) Solve for x_i's.

$$x_1^2 = y_2(\underline{x_1^2 + x_2^2}) = y_1 y_2$$

$$x_2^2 = y_1 - x_1^2 = y_1 - y_1 y_2$$

$$= y_1(1 - y_2)$$



$$0 < X_1^2 + X_2^2 < 1$$

In (I) : $x_1 = \sqrt{y_1 y_2}$ $\quad x_2 = \sqrt{y_1(1-y_2)}$

In (II) : $x_1 = -\sqrt{y_1 y_2}$ $\quad x_2 = \sqrt{y_1(1-y_2)}$

In (III) : $x_1 = -\sqrt{y_1 y_2}$ $\quad x_2 = -\sqrt{y_1(1-y_2)}$

In (IV) : $x_1 = \sqrt{y_1 y_2}$ $\quad x_2 = -\sqrt{y_1(1-y_2)}$

For $D_{\vec{y}}$ $\quad 0 < x_1^2 + x_2^2 < 1$

$$0 < y_1 y_2 + y_1 - y_1 y_2 < 1$$

$$0 < y_1 < 1$$

$$\begin{cases} 0 < x_1^2 + x_2^2 < 1 \\ 0 < x_1^2 < 1 \\ 0 < y_1 y_2 < 1 \\ 0 < y_2 < 1 \end{cases}$$

④ Find J and $|J|$

In I:

$$J = \begin{array}{c|cc|} & y_1 & y_2 \\ x_1 & \frac{1}{2}\sqrt{\frac{y_2}{y_1}} & \frac{1}{2}\sqrt{\frac{y_1}{y_2}} \\ x_2 & \frac{1}{2}\sqrt{\frac{1-y_2}{y_1}} & -\frac{1}{2}\sqrt{\frac{y_1}{1-y_2}} \end{array}$$

$$= -\frac{1}{4}\sqrt{\frac{y_2}{1-y_2}} - \frac{1}{4}\sqrt{\frac{1-y_2}{y_2}} = -\frac{1}{4}\sqrt{\frac{y_2^2}{y_2(1-y_2)}} - \frac{1}{4}\sqrt{\frac{(1-y_2)^2}{y_2(1-y_2)}}$$

$$= \frac{-1}{4\sqrt{y_2(1-y_2)}} \qquad |J| = \frac{1}{4\sqrt{y_2(1-y_2)}}$$

In (II):

$$J = \begin{vmatrix} \frac{1}{2}\sqrt{\frac{y_2}{y_1}} & \frac{1}{2}\sqrt{\frac{y_1}{y_2}} \\ -\frac{1}{2}\sqrt{\frac{1-y_2}{y_1}} & \frac{1}{2}\sqrt{\frac{y_1}{1-y_2}} \end{vmatrix}$$

$$= \frac{1}{4\sqrt{y_2(1-y_2)}} = |J|$$

④ Find $f(y_1, y_2)$

$$= \begin{cases} \frac{1}{\pi} \cdot \frac{1}{4\sqrt{y_2(1-y_2)}}, & 0<y_1<1, \; 0<y_2<1 \\ 0, & \text{o.w.} \end{cases}$$

$$= f_{\vec{X}}(x_1, x_2) \cdot |J|$$

$$= f_{\vec{X}}(y_1 y_2, \; y_1(1-y_2)) \, |J|$$

ex) Let X_1, X_2, X_3 be indep RVs such that

$$p(x_i) = \frac{\mu_i^{x_i} e^{-\mu_i}}{x_i!}, \quad x_i = 0, 1, 2, \ldots$$

Let $Y = X_1 + X_2 + X_3$

Find MGF of Y.

$$M_Y(t) = \mathbb{E}[e^{tY}] = \mathbb{E}[e^{t(X_1+X_2+X_3)}]$$

$$= \mathbb{E}[e^{tX_1} e^{tX_2} e^{tX_3}]$$

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indep.

$$= \mathbb{E}[e^{tX_1}]\,\mathbb{E}[e^{tX_2}]\,\mathbb{E}[e^{tX_3}]$$

$$= M_{X_1}(t)\,M_{X_2}(t)\,M_{X_3}(t)$$

$$M_{X_i}(t) = e^{\mu_i(e^t-1)}$$

$$= e^{\mu_1(e^t-1)}\,e^{\mu_2(e^t-1)}\,e^{\mu_3(e^t-1)}$$

$$= e^{(e^t-1)\boxed{(\mu_1+\mu_2+\mu_3)}}$$

μ_i

$$\cancel{P(y_1,y_2,y_3) = \frac{(\mu_1+\mu_2+\mu_3)\,e^{-(\mu_1+\mu_2+\mu_3)}}{y_1!\,y_2!}}$$

$$\cancel{= \frac{\mu_1\,\mu_2\,\mu_3\,e^{-(\mu_1+\mu_2+\mu_3)}}{y_1!\,y_2!\,y_3!}}$$

$$P(y) = \frac{(\mu_1+\mu_2+\mu_3)^{y}\,e^{-(\mu_1+\mu_2+\mu_3)}}{y!}$$