Stat 401 - 11/6/17

Sec 3.3 - The Γ, χ^2, and β Distributions

(1) Exponential and Gamma Dist

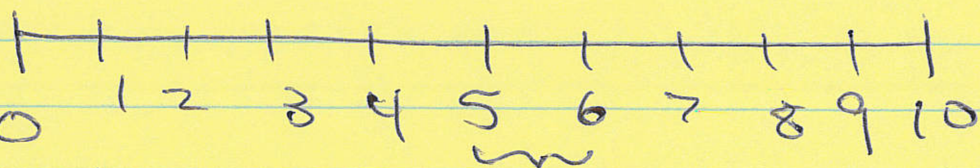
Poisson Dist: # outcomes in a given time interval.

We can look at waiting time between outcomes/events

ex) 10 calls/hr at a helpdesk. We are interested in a 1 hour timeperiod $= w$

$\lambda = 10$

$\lambda w = 10 = \mu$ ← for Poisson.



waiting time from 0 calls to 1 call

waiting time between calls 5 and 6.

can be different.

this is a RV (continuous).

def) Exponential Dist.

Define RV X to be

X = waiting time for a Poisson Process until the 1^{st} change occurs.

X is continuous

$x > 0$
waiting time is positive

The CDF of X:

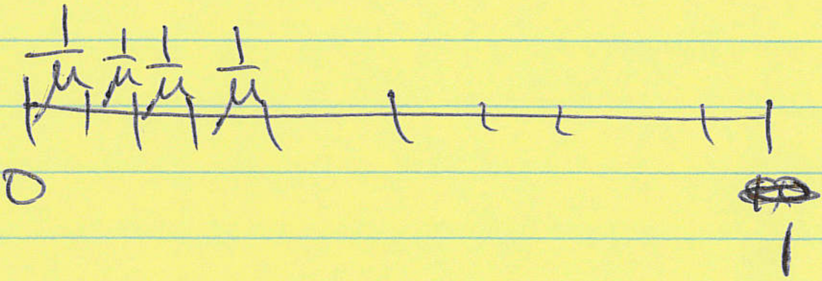
$$F_X(x) = \mathbb{P}[X \le x]$$

$$= \begin{cases} 1 - e^{-\mu x}, & x > 0 \\ 0, & \text{o.w.} \\ & (x \le 0) \end{cases}$$

The PDF of X:

$$f_X(x) = \begin{cases} \mu e^{-\mu x}, & x > 0 \\ 0, & \text{o.w.} \end{cases}$$

Let $\beta = \frac{1}{\mu}$



PDF of X:

$$f_X(x) = \begin{cases} \frac{1}{\beta} e^{-\frac{x}{\beta}} & , \; x > 0 \\ 0 & , \; \text{o.w.} \end{cases}$$

$\mu = \mathbb{E}[X] = \beta$

$\sigma^2 = \text{Var}[X] = \beta^2$

We say $X \sim \text{Exp}(\beta)$

MGF $M_X(t) = \frac{1}{1-\beta t}$ for $t < \frac{1}{\beta}$

def] Gamma Dist.

Define RV X to be

X = waiting time ~~for~~ until the α^{th} change occurs.

Let $\beta = \frac{1}{\mu}$ ← rate from Poisson.

The PDF of X is

$$f_X(x) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}}, & x>0 \\ 0, & \text{o.w.} \end{cases}$$

where $\alpha > 0$, $\beta > 0$

We say $X \sim \Gamma(\alpha, \beta)$

$\Gamma(\alpha)$ is a Gamma Function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\,dx \text{ for } \alpha > 0$$

$\Gamma(1) = 1$

$\Gamma(\alpha) = (\alpha - 1)!$ if $\alpha \in \mathbb{Z}^+$ and $\alpha > 1$

Find using Int. by parts.

MGF for X

$$M_X(t) = \mathbb{E}[e^{tX}]$$

$$= \int_0^\infty e^{tx} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}} dx$$

$$= \int_0^\infty \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x\left(\frac{1}{\beta} - t\right)} dx$$

$\frac{1-\beta t}{\beta}$

Let $y = \frac{1-\beta t}{\beta} x$

$\Rightarrow \quad x = \frac{y\beta}{1-\beta t} \Rightarrow \frac{\beta}{1-\beta t} dy = dx$

If $x = 0$, $y = 0$
If $x = \infty$, $y = \infty$

restriction on t!

$1 - \beta t > 0$

$t < \frac{1}{\beta}$

⑥

$$= \int_0^{\infty} \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \left(\frac{\beta}{1-\beta t}\right)^{\alpha-1} y^{\alpha-1} e^{-y} \frac{\beta}{1-\beta t} dy$$

$$= \int_0^{\infty} \frac{1}{\cancel{\beta^{\alpha}}\Gamma(\alpha)} \frac{\cancel{\beta^{\alpha-1}}}{(1-\beta t)^{\alpha-1}} y^{\alpha-1} e^{-y} \frac{\cancel{\beta}}{1-\beta t} dy$$

$$= \frac{1}{(1-\beta t)^{\alpha}} \underbrace{\int_0^{\infty} \frac{1}{1^{\alpha}\Gamma(\alpha)} y^{\alpha-1} e^{-\frac{y}{1}} dy.}_{\text{If } \beta = 1}$$

$= 1$ by PDF properties

$$= \frac{1}{(1-\beta t)^{\alpha}} \quad \text{for } t < \frac{1}{\beta}$$

$$\mathbb{E}[X] = \alpha\beta$$

$$\mathrm{Var}[X] = \alpha\beta^2$$

<u>Thm</u> Let $X_1, \ldots, X_n$ be indep RVs

Let $X_i \sim \Gamma(\alpha_i, \beta)$

Let $Y = \sum_{i=1}^{n} X_i$

then $Y \sim \Gamma\left(\sum_{i=1}^{n} \alpha_i, \beta\right)$

Proof by MGF

(2) Chi-Square Dist.

→ special case of Gamma where

$\alpha = \frac{r}{2}$ and $\beta = 2$

$r \in \mathbb{Z}^+$

The PDF is :

$$f_X(x) = \begin{cases} \frac{1}{2^{r/2}\,\Gamma\left(\frac{r}{2}\right)} x^{\frac{r}{2}-1} e^{-\frac{x}{2}}, & x > 0 \\ 0, & \text{o.w.} \end{cases}$$

$\mathbb{E}[X] = r$

$Var[X] = \alpha\beta^2 = \frac{r}{2} \cdot 2^2 = 2r$

MGF: $M_X(t) = (1-2t)^{-\frac{r}{2}}$, $t < \frac{1}{2}$

r = degrees of freedom

we say $X \sim \chi^2(r)$

ex) $\cancel{1}$ $\cancel{2}$ 3 $\cancel{4}$

Pick #1 : 2

Pick #2 : 4

Pick #3 : 1

Pick #4 : 3 ← chosen for me
No choice.
No freedom to choose.

df = degrees of freedom = 3

3 Beta Distribution

Why use?

- defined on interval from 0 to 1
  bounded.

- Scale or shift to any finite interval of interest.

Where
- PERT (program evaluation + review technique).
- critical path methods
- Joint cost schedule modeling
- Bayesian Inference

  ⋮
- order statistics.

How to Derive this Dist?

- start w/ 2 indep Gamma RVs, then transform.

$$\left.\begin{array}{l} X_1 \sim \Gamma(\alpha, 1) \\ X_2 \sim \Gamma(\beta, 1) \end{array}\right\} \text{indep.}$$

$$f(x_1, x_2) = \begin{cases} \frac{1}{\Gamma(\alpha) 1^\alpha} x_1^{\alpha-1} e^{-\frac{x_1}{1}} \frac{1}{\Gamma(\beta) 1^\beta} x_2^{\beta-1} e^{-x_2} & \text{for } x_1 > 0 \quad x_2 > 0 \\ 0, & \text{o.w.} \end{cases}$$

by indep.

Let $Y_1 = X_1 + X_2$

$$Y_2 = \frac{X_1}{X_1 + X_2}$$

Step 1: Find pdf of (Y_1, Y_2).

$y_2(x_1 + x_2) = x_1 = y_1 y_2$

$y_1 = x_1 + x_2 = \quad x_2 = y_1 - y_1 y_2$
$= y_1(1 - y_2)$

$J = -y_1$

$D_{\underset{\sim}{y}}$: $0 < y_1 < \infty$

$0 < y_2 < 1$

$|J| = y_1$

$= |-y_1|$

$$f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} \frac{1}{\Gamma(\alpha)\Gamma(\beta)} y_1^{\alpha+\beta-1} y_2^{\alpha-1} (1-y_2)^{\beta-1} e^{-y_1} & \text{for } \begin{array}{l} 0 < y_1 < \infty \\ 0 < y_2 < 1 \end{array} \\ 0, & \text{o.w.} \end{cases}$$

Step 2: Find marginal pdf for Y_2.

$$\frac{y_2^{\alpha-1} (1-y_2)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \underbrace{\int_0^\infty y_1^{(\alpha+\beta)-1} e^{-y_1} \, dy_1}_{\text{Gamma Funct}}$$

$= \Gamma(\alpha+\beta)$

Beta Dist

$$f_X(x) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & 0 < x < 1 \\ 0, & \text{o.w.} \end{cases}$$

$$\mathbb{E}[X] = \frac{\alpha}{\alpha+\beta} \qquad Var[X] = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$$