Stat 401 - 11/13/17

Hw: Sec 3.3 # 1, 19a, 24
Sec 3.4 # 2, 3, 4, 10, 28
Sec 3.5 # 1
Sec 3.6 # 1.
due 11/20/17

Final Exam - Friday, December 15
from 10:30 - 12:30
Same classroom SES 138

Sec 3.4 - Normal Dist.

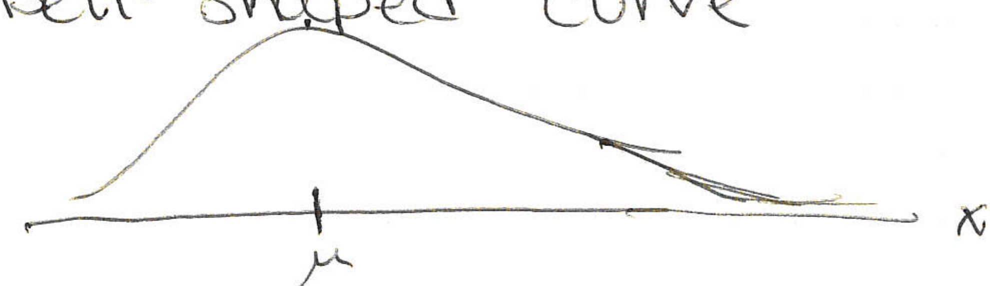
↓

Gaussian Dist.

Properties

(0) Continuous for $-\infty < x < \infty$

(1) Bell-shaped curve

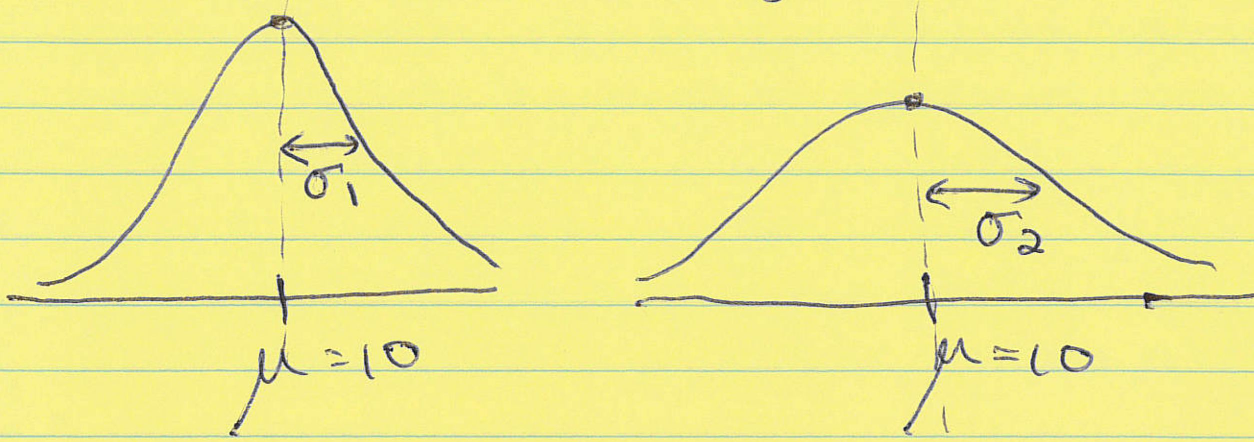


(2) center is at the mean

$\mu = \mathbb{E}[X]$

Location parameter.

3) The spread of curve is determined by std dev / variance

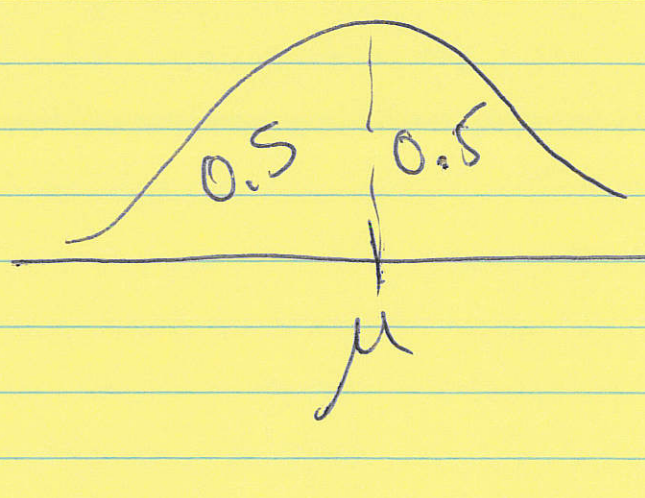


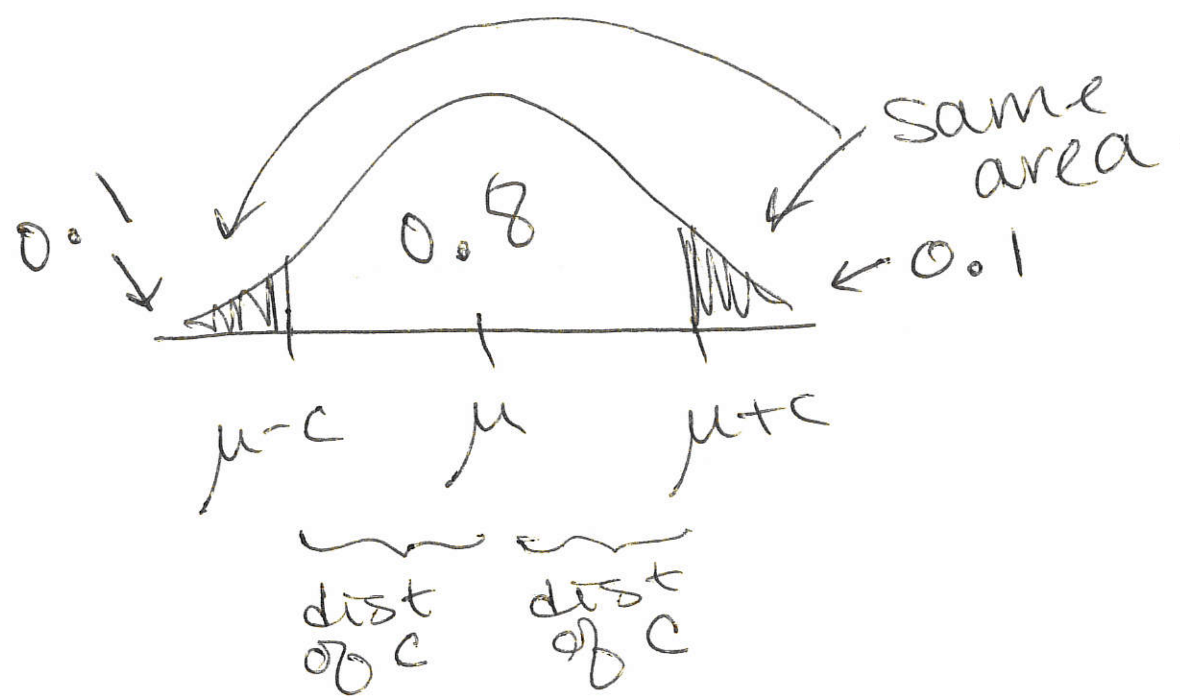
where $\sigma_1 < \sigma_2$

σ or σ^2 is called the scale parameter.

4) The entire area under curve $= 1$

5) Curve is symmetric about the mean!





def Standard Normal Dist.

A RV Z is said to be std. normal with mean $= 0$ and variance $= 1$ if its pdf is given by

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad -\infty < z < \infty$$

We write $Z \sim N(0, 1)$

mean variance

MGF:
$$M_Z(t) = \mathbb{E}\left[e^{tZ}\right]$$
$$= e^{t^2/2}, \quad -\infty < t < \infty$$

Define the CDF of Z by

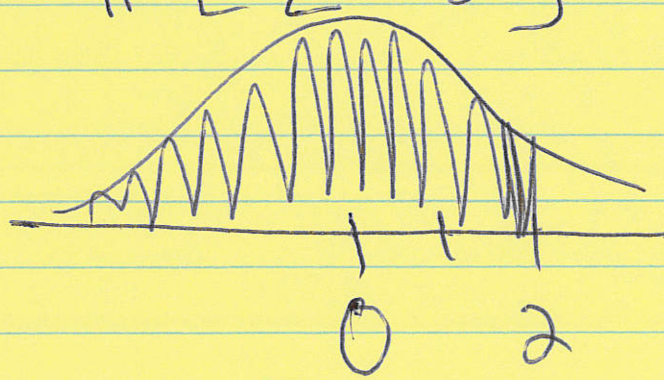
$$F(z) = \mathbb{P}[Z \leq z]$$
$$= \Phi(z)$$

ex) $\mathbb{P}[Z \leq 2] = \Phi(2)$

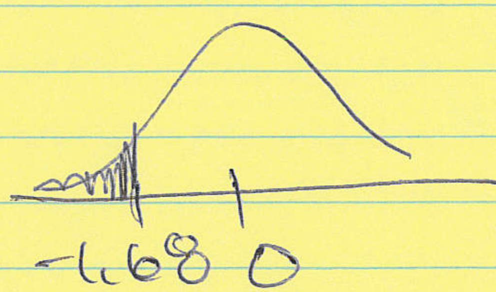
Finding Probabilities

- Table gives prob. to the left of # of interest.

ex) $\mathbb{P}[Z < 2] = \Phi(2)$
$= 0.9772$

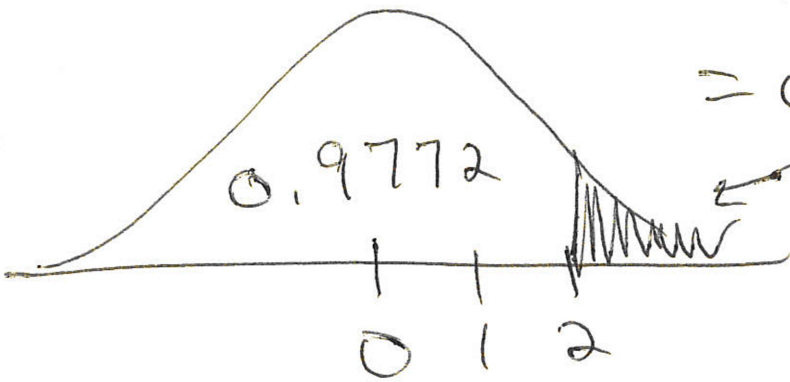


ex) $\mathbb{P}[Z < -1.68] = 0.0465$



$= \mathbb{P}[Z < -2]$

$$\mathbb{P}[Z > 2] = 1 - \mathbb{P}[Z \le 2]$$
$$= 1 - \Phi(2)$$
$$= 1 - 0.9772$$
$$= 0.0228$$

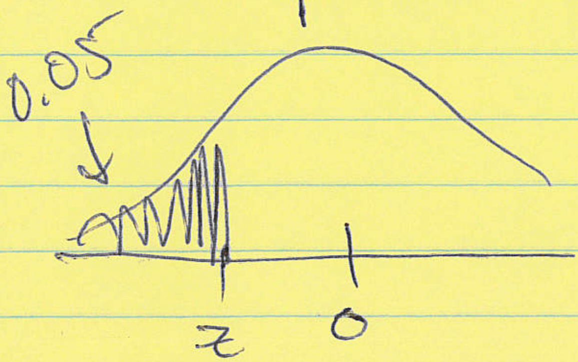


$$\mathbb{P}[-1 < Z < 2]$$

$$= \mathbb{P}[Z < 2] - \mathbb{P}[Z < -1]$$
$$= \Phi(2) - \Phi(-1)$$
$$= 0.9772 - 0.1587$$
$$= 0.8185$$

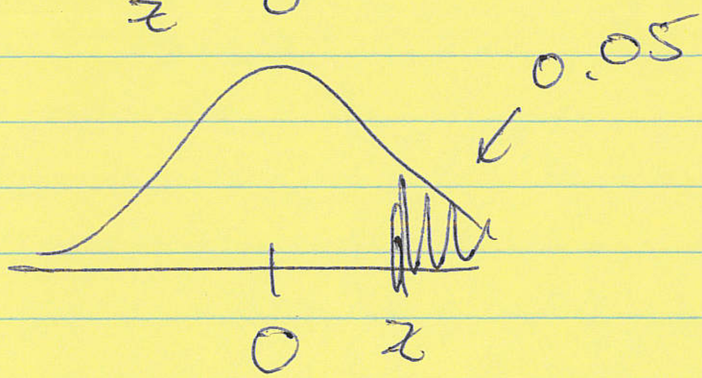
Finding z-values from prob.

ex) Find z-value where prob. to left is 0.05



$\Phi(z) = 0.05$

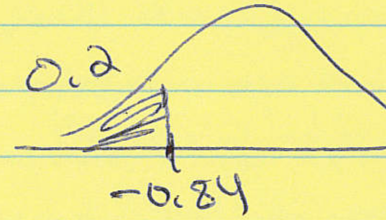
$z = -1.645$



$z = +1.645$

ex) Find z-value where prob to right is 0.2

=



$\Phi(z) = 0.8$

$z = 0.84$

Remark)

- $\phi(-z) = 1 - \phi(z)$
  by symmetry.
- $\phi(z_p) = p$

  p = area under curve to the left of z.

  $z_{0.05} = -1.645$

  $z_{0.8} = 0.84$

  $z_{0.2} = -0.84$

Consider the transformation

$$X = bZ + a \quad \text{where } b > 0.$$

Note $Z = \dfrac{X-a}{b}$

Jacobian $= \dfrac{1}{b}$ and $|J| = \dfrac{1}{b}$

The pdf of X is

$$f_X(x) = \frac{1}{b\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2}, \quad -\infty < x < \infty$$

By MGF, $E[X] = a = \mu$

$Var[X] = b^2 = \sigma^2$

def) We say a RV X has a Normal Dist with mean μ and variance σ^2 if its pdf is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

for $-\infty < x < \infty$

We say $X \sim N(\mu, \sigma^2)$

The MGF is given by

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

for $-\infty < t < \infty$

ex) If X has MGF

$$M(t) = e^{2t + 32t^2}$$

identify the dist and its parameters.

SOL

$$M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2} \leftarrow \text{Normal}$$

$$M(t) = e^{2t + 32t^2}$$

$$\mu = 2 \quad \text{and} \quad \frac{1}{2}\sigma^2 = 32$$

$$\sigma^2 = 64$$

$$X \sim N(2, 64)$$

Finding Prob.

We have a transformation -

$$z = \frac{X - \mu}{\sigma} \quad \text{where } X \sim N(\mu, \sigma^2)$$

$$Z \sim N(0, 1).$$

All converted values called z-scores.

$$z = \frac{x-\mu}{\sigma}$$

ex) $X \sim N(2, 25)$

Find $\mathbb{P}[\,0 < X < 10\,]$

Sol) $\mu = 2 \quad \sigma^2 = 25 \quad \sigma = 5$

$$z_1 = \frac{0-2}{5} = -0.4$$

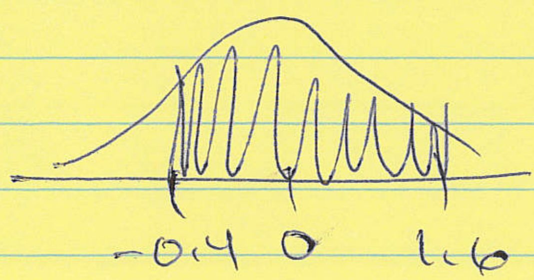
$$z_2 = \frac{10-2}{5} = 1.6$$

$$\mathbb{P}[0 < X < 10] = \mathbb{P}[-0.4 < Z < 1.6]$$

$$= \Phi(1.6) - \Phi(-0.4)$$

$$= 0.9452 - 0.3446$$

$$= 0.6006$$



Thm) Let $X_1, \dots, X_n$ be indep. RVs such that

$$X_i \sim N(\mu_i, \sigma_i^2) \text{ for } i = 1, \dots, n$$

Let $Y = \sum_{i=1}^{n} a_i X_i$ then

a_i = constant

$$Y \sim N\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right)$$

<u>Corollary</u>: Let $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$

Let $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$

Then $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

<u>Thm</u>: If $X \sim N(\mu, \sigma^2)$ and $\sigma^2 > 0$

then the RV

$$V = \frac{(X-\mu)^2}{\sigma^2} \sim \chi^2(1)$$

<u>Sketch of proof</u>:

If $V = \frac{(X-\mu)^2}{\sigma^2} = \left(\frac{X-\mu}{\sigma}\right)^2 = Z^2$

where $Z \sim N(0,1)$.

CDF for V is

$$F(v) = \mathbb{P}[Z^2 \le v]$$
$$= \mathbb{P}[-\sqrt{v} \le Z \le \sqrt{v}]$$

Find PDF using 1st deriv.

...

$$f(v) = \frac{1}{\Gamma\left(\frac{1}{2}\right) 2^{1/2}} v^{\frac{1}{2}-1} e^{-\frac{v}{2}}$$

PDF

for $0 \leq v < \infty$

$r = 1$

so $V \sim \chi^2(1)$.