Stat 401 - 11/15/17

Sec 3.5 - Multivariate Normal Dist.

def) Multivariate Normal with mean $\vec{0}$ and covariance I_n

vector of 0's

identity

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}$$

Recall $Z \sim N(0,1)$ $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

$z \in \mathbb{R}$

Suppose $Z_1, \ldots, Z_n \overset{iid}{\sim} N(0,1)$

Then $\vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}_{n\times 1} = (Z_1, \ldots, Z_n)'$

random vector.

transpose

is jointly normal.

Joint pdf is:

$$f_{\vec{Z}}(z_1, \ldots, z_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{z_i^2}{2}}$$

$$= (2\pi)^{-n/2} e^{-\frac{1}{2}\sum_{i=1}^{n} z_i^2}$$

$$e^{-\frac{z_1^2}{2} - \frac{z_2^2}{2} - \cdots}$$

$$= (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2}\vec{z}'\vec{z}}$$

$$\vec{z}'\vec{z} = (z_1, \ldots, z_n)\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = z_1^2 + z_2^2 + \cdots + z_n^2$$

$$\mathbb{E}[\vec{Z}] = \vec{0}$$

$$\mathbb{E}\begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix} = \begin{bmatrix} \mathbb{E}(Z_1) \\ \vdots \\ \mathbb{E}(Z_n) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$cov(Z_1, Z_1)$
$= Var(Z_1)$
$= 1$

$$Cov(\vec{Z}) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = I_n$$

$cov(Z_1, Z_2)$
$= 0$

We say $\vec{Z} \sim N_n(\vec{0}, I_n)$

\# of RVs

MGF of $\vec{Z}$ is

$$M_{\vec{Z}}(\vec{t}) = \mathbb{E}\left[e^{\vec{t}\,'\vec{Z}}\right]$$

$$= \mathbb{E}\left(e^{(t_1, t_2, \ldots, t_n)\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}}\right)$$

$$= e^{\frac{1}{2}\vec{t}\,'\vec{t}}$$

<u>ex)</u> Bivariate ~~B~~ Normal

$$\vec{Z} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \sim N_2(\vec{0}, I_2)$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} Var(Z_1) & Cov(Z_1, Z_2) \\ Cov(Z_2, Z_1) & Var(Z_2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

def) We say $\vec{X} = (X_1, X_2)'$ is bivariate normal if

$$\vec{X}_{2\times1} = \vec{A}_{2\times2}\,\vec{Z}_{2\times1} + \vec{\mu}_{2\times1}$$

i.e.

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

$$\Rightarrow \quad X_1 = aZ_1 + bZ_2 + \mu_1$$
$$X_2 = cZ_1 + dZ_2 + \mu_2$$

Remarks)

1. $$\mathbb{E}[\vec{X}] = \begin{pmatrix} \mathbb{E}[X_1] \\ \mathbb{E}[X_2] \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

2a. $$\mathrm{Var}[X_1] = a^2 + b^2$$

proof)
$$\begin{aligned} \mathrm{Var}[X_1] &= \mathrm{Var}[aZ_1 + bZ_2 + \mu_1] \\ &= \mathrm{Var}[aZ_1 + bZ_2] \\ &\overset{\text{indep}}{=} a^2\,\mathrm{Var}[Z_1] + b^2\,\mathrm{Var}[Z_2] \\ &= a^2 + b^2 \end{aligned}$$

2b $\text{var}[X_2] = c^2 + d^2$

2c $\text{Cov}(X_1, X_2)$

$$= \text{Cov}(aZ_1 + bZ_2,\ cZ_1 + dZ_2)$$

$$= \mathbb{E}\left[\underbrace{(aZ_1 + bZ_2)(cZ_1 + dZ_2)}_{acZ_1^2 + adZ_1Z_2 + bcZ_1Z_2 + bdZ_2^2}\right]$$

$$= ac + bd$$

2d Define $\Sigma = AA'$ = (variance-) covariance matrix.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$AA' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$= \begin{pmatrix} a^2 + b^2 & ac+bd \\ ca+db & \underline{c^2+d^2} \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_1^2 & \text{Cov}(X_1, X_2) \\ \text{Cov}(X_1, X_2) & \sigma_2^2 \end{pmatrix}$$

$$\rho = \frac{Cov(X_1, X_2)}{\sigma_1 \sigma_2}$$

$$= \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

always symmetric

A $\underbrace{\text{positive semidefinite}}$ matrix.

- all eigenvalues are nonneg.
- $x' \Sigma x \geq 0 \quad \forall x \in \mathbb{R}^n$

2e The pdf of $\vec{X} = (X_1, X_2)'$ is given by.

$$f_{\vec{X}}(\vec{x}) = \frac{1}{2\pi |\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\vec{x}-\vec{\mu})' \Sigma^{-1} (\vec{x}-\vec{\mu})\right\}$$

determinant

$$M_{\vec{X}}(\vec{t}) = e^{t'\vec{\mu} + \frac{1}{2}\vec{t}'\Sigma t}$$

$$f(x_1, x_2) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 \right] \right\}$$

$$|\Sigma|^{1/2}$$

$$\det(\Sigma) = \det\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$
$$= \sigma_1^2 \sigma_2^2 (1-\rho^2)$$

We can say that

$$\vec{X} = (X_1, X_2)' \sim N_2(\vec{\mu}, \Sigma)$$

OR.

$$(X_1, X_2)' \sim N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$$

③ $\vec{\mu}$, Σ uniquely determines the dist of $\vec{X}$

4 We could have defined

$$X_1 = a_1 Z_1 + a_2 Z_2 + a_3 Z_3 + \mu_1$$
$$X_2 = b_1 Z_1 + b_2 Z_2 + b_3 Z_3 + \mu_2.$$

but using linear algebra, equivalent to

$$X_1 = \vec{a}_1 \vec{Z}_1 + \vec{a}_2 \vec{Z}_2 + \mu_1$$
$$X_2 = \vec{b}_1 \vec{Z}_1 + \vec{b}_2 \vec{Z}_2 + \mu_2$$

5 Marginal Dist

$$X_1 \sim N(\mu_1, a^2+b^2) = N(\mu_1, \sigma_1^2)$$
$$X_2 \sim N(\mu_2, c^2+d^2) = N(\mu_2, \sigma_2^2)$$