Stat 401 - 11/17/2017

Sec 3.5 continued.

Remarks!

5. Marginal Distributions

$$X_1 \sim N(\mu_1, \sigma_1^2)$$

$$X_2 \sim N(\mu_2, \sigma_2^2)$$

6. If $\vec{X} = (X_1, X_2)' \sim N_2(\vec{\mu}, \Sigma)$ and $\rho = 0$, then X_1 and X_2 are independent.

proof) The pdf of $\vec{X}$ when $\rho = 0$ is

$$f_{\vec{X}}(\vec{x}) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-0^2}} \exp\left\{-\frac{1}{2(1)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2(0) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]\right.$$

$$= \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{1}{2}\frac{(x_1-\mu_1)^2}{\sigma_1^2}} \cdot \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{1}{2}\frac{(x_2-\mu_2)^2}{\sigma_2^2}}$$

$$= f(x_1) \cdot f(x_2)$$

$\therefore$ Independent.

7 Conditional Dist of X_1 given $X_2 = x_2$. is

$$X_1 \mid X_2 = x_2 \sim N\Big(\underbrace{\mu_1 + \frac{\sigma_1}{\sigma_2}\rho(x_2 - \mu_2)}_{\text{mean}}, \underbrace{\sigma_1^2(1-\rho^2)}_{\text{variance}}\Big)$$

Conditional Dist of X_2 given $X_1 = x_1$

$$X_2 \mid X_1 = x_1 \sim N\Big(\underbrace{\mu_2 + \frac{\sigma_2}{\sigma_1}\rho(x_1 - \mu_1)}_{\text{mean}}, \underbrace{\sigma_2^2(1-\rho^2)}_{\text{variance}}\Big)$$

$$\mathbb{E}[X_2 \mid X_1 = x_1] = \mu_2 + \frac{\sigma_2}{\sigma_1}\rho(x_1 - \mu_1)$$

Find $\mathbb{P}[X_2 = 3 \mid X_1 = x_1]$

$\rightarrow$ Find z-score

Thm) Suppose $\vec{X} \sim N_n(\vec{\mu}, \Sigma)$

If $n=2$, then $\vec{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, $\vec{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$

and $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$

Assume Σ is positive definite, then the dist of

$$\vec{X}_1 \mid \vec{X}_2 = \vec{x}_2 \quad \text{is}$$

$$N_m\left(\vec{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\vec{X}_2 - \vec{\mu}_2),\ \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right)$$

Sec 3.6 - t and F distributions.

① t - distribution.

Let $Z \sim N(0,1)$ and

$V \sim \chi^2(r)$

Suppose that Z and V are independent then

$$T = \frac{Z}{\sqrt{\frac{V}{r}}}$$ follows a t - dist with r degrees of freedom.

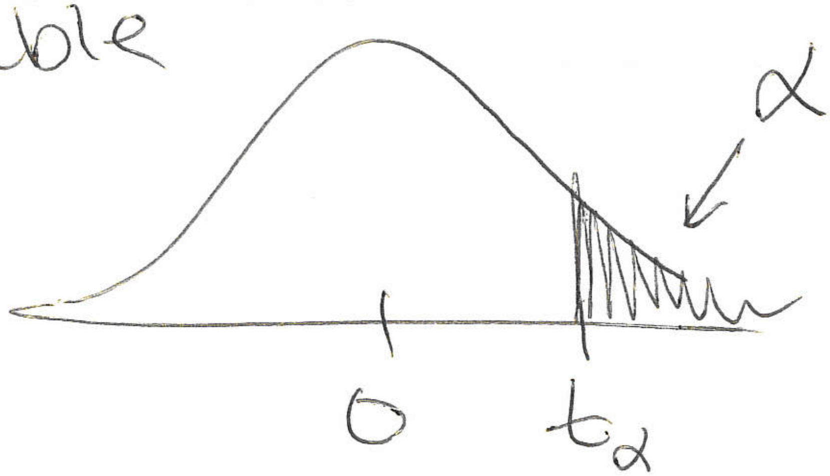
We say $T \sim t(r)$

Properties

1. Looks like a bell-shaped curve.
2. symmetric
3. center at 0
4. As $r \to \infty$, it becomes more and more like a normal dist.

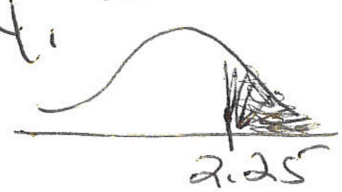
To calculate

Table



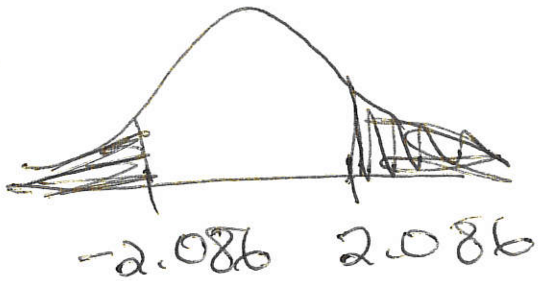
ex) Find $\mathbb{P}[t > 2.25]$ where $r = 24$.

$\in (0.015, 0.02)$



ex) $P[|T| > 2.086]$ with 20 df.

$\simeq 2(0.025) = 0.05$



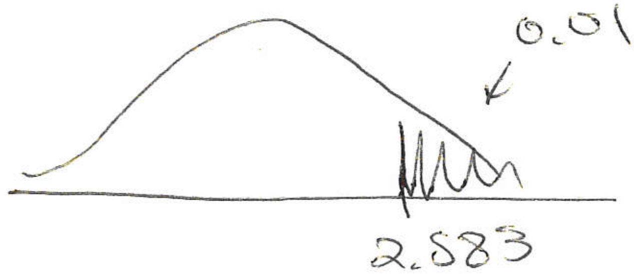
ex) Find k s.t. $P[T > k] = 0.01$ with 16 df.

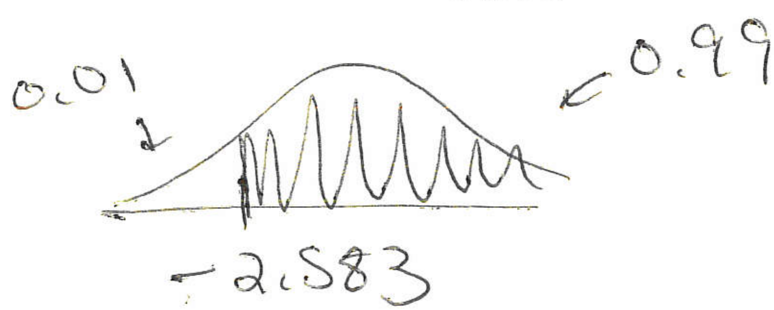
$k = 2.583$

i.e. $t_{0.01}(16) = 2.583$

$\uparrow$ area to right.

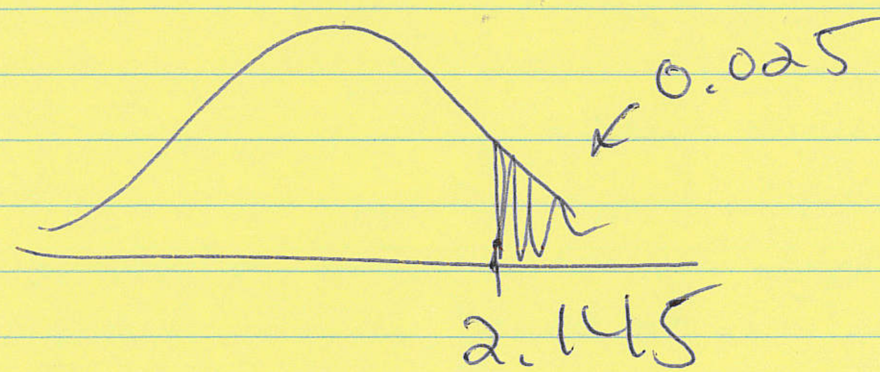
Note: $t_{0.01}(16) = -t_{0.99}(16)$





ex) Find value of $t_{0.975}(14)$

$t_{0.025}(14) = 2.145$



Derivation of Dist

Since Z and V are indep

$$f(z,v) = f(z)f(v)$$

$$= \begin{cases} \dfrac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \dfrac{1}{\Gamma(\frac{r}{2}) 2^{r/2}} v^{\frac{r}{2}-1} e^{-\frac{v}{2}} & \begin{array}{c} -\infty < z < \infty \\ 0 < v < \infty \end{array} \\ 0 \quad , & \text{o.w.} \end{cases}$$

Define a new RV T to be

$$T = \frac{Z}{\sqrt{V/r}}$$

Use change of variable technique.

Let $t = \dfrac{z}{\sqrt{v/r}}$ and $u = v$

$$\Rightarrow \quad z = t\sqrt{\frac{u}{r}} \text{ and } v = u$$

$$\text{So } J = \begin{vmatrix} \frac{\partial z}{\partial t} & \frac{\partial z}{\partial u} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial u} \end{vmatrix} = \frac{\sqrt{u}}{\sqrt{r}} = |J|$$

If $0 < v < \infty \Rightarrow 0 < u < \infty$

if $-\infty < z < \infty \Rightarrow -\infty < t < \infty$

thus

$$g(t,u) = \begin{cases} \dfrac{1}{\sqrt{2\pi}\,\Gamma\left(\frac{r}{2}\right) 2^{r/2}} u^{\frac{r}{2}-1} e^{-\frac{1}{2}\frac{ut^2}{r} - \frac{u}{2}} \dfrac{\sqrt{u}}{\sqrt{r}} & \text{for } -\infty < t < \infty,\ 0 < u < \infty \\ 0 & \text{, o.w.} \end{cases}$$

the pdf for T is

$$\frac{\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{\pi r}\;\Gamma\left(\frac{r}{2}\right)} \cdot \frac{1}{\left(1+\frac{t^2}{r}\right)^{\frac{r+1}{2}}}$$

for $-\infty < t < \infty$

$$\mathbb{E}[T^k] = \mathbb{E}[Z^k] \cdot \frac{2^{-\frac{k}{2}}\,\Gamma\left(\frac{r}{2}-\frac{k}{2}\right)}{\Gamma\left(\frac{r}{2}\right) r^{-k/2}}$$

if $k < r$

$$\mathrm{Var}[T] = \frac{r}{r-2}$$

when $k=1$, $\mathbb{E}[T] = 0$

t-dist w/ $r > 2$ df has mean of 0 and variance $\frac{r}{r-2}$.

② F - dist.

def) Let $U \sim \chi^2(r_1)$ and

$V \sim \chi^2(r_2)$

Suppose U and V are indep.

then $F = \dfrac{U/r_1}{V/r_2}$ follows an

F-dist with r_1 and r_2 d.f.

we say $F \sim F(r_1, r_2)$

The pdf of F is

$$f(w) = \begin{cases} \dfrac{\Gamma\left(\frac{r_1+r_2}{2}\right)\left(\frac{r_1}{r_2}\right)^{r_1/2}}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right)} \dfrac{w^{\frac{r_1}{2}-1}}{\left(1+\frac{r_1 w}{r_2}\right)^{\frac{r_1+r_2}{2}}} & \text{for } 0 < w < \infty \\ 0, & \text{o.w.} \end{cases}$$

The k^{th} moment of F is

$$\mathbb{E}[F^k] = \left(\frac{r_2}{r_1}\right)^k \mathbb{E}[U^k]\,\mathbb{E}[V^{-k}]$$

$$\mathbb{E}[F] = \frac{r_2}{r_2-2}$$

$$= \frac{r_2}{r_1} \underbrace{\mathbb{E}[U]\,\mathbb{E}[V^{-1}]}_{r_1 \; \frac{2^{-1}\,\Gamma\left(\frac{r_2}{2}-1\right)}{\Gamma\left(\frac{r_2}{2}\right)}}$$

__Thm__ (Student)

Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

Define RVs $\bar{X} = \frac{1}{n}\sum X_i$ and

$S^2 = \frac{1}{n-1}\sum (X_i - \bar{X})^2$ then

a) $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

b) $\bar{X}$ and S^2 are indep.

c) $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$

d) $T = \frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t(n-1)$