Stat 401- 11/20/17

Sec 3.7 #1 - Find PDF and mean only
#3 - Find mean and variance only
Sec 5.1 # 2, 3, 5

} due 11/27/17

Sec 3.7 - Mixture of Distributions

Case 1:

Let	$X_1, \dots, X_k$	be RVs
pdf	$f_1, \dots, f_k$	
mean	$\mu_1, \dots, \mu_k$	
variances	$\sigma_1^2, \dots, \sigma_k^2$	

Let $p_1, \dots, p_k$ are positive mixing probabilities where $p_1 + p_2 + \cdots + p_k = 1$

Consider the function

$$f(x) = p_1 f_1(x) + p_2 f_2(x) + \cdots + p_k f_k(x)$$
$$= \sum_{i=1}^{k} p_k f_k(x)$$

$f(x)$ is a pdf of a new continuous RV X.

The expected value of X is:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x \cdot \sum_{i=1}^{k} p_i f_i(x)\, dx$$

$$= \int_{-\infty}^{\infty} \sum_{i=1}^{k} x\, p_i f_i(x)\, dx$$

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$$= \sum_{i=1}^{k} p_i \int_{-\infty}^{\infty} x\, f_i(x)\, dx$$

$$= \sum_{i=1}^{k} p_i \mu_i$$

$= \bar{\mu}$ ← weighted average of $\mu_1, \mu_2, \ldots, \mu_k$

The variance:

$$\text{Var}[X] = \mathbb{E}\left[(X - \bar{\mu})^2\right]$$

$$= \int_{-\infty}^{\infty} (x - \bar{\mu})^2 f(x)\, dx$$

$$= \int (x-\bar{\mu})^2 \sum p_i f_i(x)\, dx$$

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$$= \sum p_i \int (x-\bar{\mu})^2 f_i(x)\, dx$$

$$= \sum p_i \int \Big[(x\underbrace{-\mu_i) + (\mu_i}_{\text{Add } 0} - \bar{\mu}) \Big]^2 f_i(x)\, dx$$

$$= \sum p_i \int \Big[(x-\mu_i)^2 + (\mu_i-\bar{\mu})^2 + 2(x-\mu_i)(\mu_i-\bar{\mu}) \Big] f_i(x)\, dx.$$

$$= \sum_{i=1}^{K} p_i \underbrace{\boxed{\int (x-\mu_i)^2 f_i(x)\, dx}}_{=\sigma_i^2} + \sum_{i=1}^{K} p_i \cancel{\int} (\mu_i-\bar{\mu})^2 \underbrace{\boxed{\int_{-\infty}^{\infty} f_i(x)\, dx}}_{=1} + 2 \sum_{i=1}^{K} p_i \underbrace{\int (x-\mu_i)(\mu_i-\bar{\mu}) f_i(x)\, dx}_{=0}$$

$$= \sum_{i=1}^{k} p_i \sigma_i^2 + \sum_{i=1}^{k} p_i (\mu_i - \bar{\mu})^2$$

$$= Var[X]$$

→ weighted average of k variances

+ weighted variance of the means

<u>Remark</u>) Characteristics are associated with a mixture and has Nothing to do with a linear combination of RVs.

<u>ex</u>) We say X has a <u>log gamma</u> distribution / pdf with parameters $\alpha > 0$, $\beta > 0$ if it has pdf.

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{-\left(\frac{1+\beta}{\beta}\right)} (\log x)^{\alpha - 1}, & x \geq 1 \\ 0, & \text{o.w.} \end{cases}$$

We say $X \sim \log \Gamma(\alpha, \beta)$

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$X_1 \sim \log\Gamma(\alpha_1, \beta_1)$

$X_2 \sim \Gamma(\alpha_2, \beta_2)$

take mixing prob. $\overset{\text{"}p_1}{p}$ and $\overset{\text{"}p_2}{\widetilde{1-p}}$.

What is pdf of mixture?

$\boxed{\text{Sol}}$ Recall Gamma pdf.

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\beta}}, & x \geq 0 \\ 0, & \text{o.w.} \end{cases}$$

$f(x) = p_1 f_1 + p_2 f_2.$

The pdf of mixture dist is:

$$f(x) = \begin{cases} \dfrac{1-p}{\Gamma(\alpha_2)\beta_2^{\alpha_2}} x^{\alpha_2-1} e^{-\frac{x}{\beta_2}} & 0 < x \leq 1 \\ \dfrac{p}{\Gamma(\alpha_1)\beta_1^{\alpha_1}} x^{-\left(\frac{1+\beta_1}{\beta_1}\right)} (\log x)^{\alpha_1-1} + \dfrac{1-p}{\Gamma(\alpha_2)\beta_2^{\alpha_2}} x^{\alpha_2-1} e^{-\frac{x}{\beta_2}} & 1 < x \\ 0, & \text{o.w.} \end{cases}$$

If $\beta_1 < \frac{1}{2}$ then

$$\mu = p(1-\beta_1)^{-\alpha_1} + (1-p)\alpha_2\beta_2$$

$$\sigma^2 = p\left[(1-2\beta_1)^{-\alpha_1} - (1-\beta_1)^{-2\alpha_1}\right] + (1-p)\alpha_2\beta_2^2 + p(1-p)\left[(1-\beta_1)^{-\alpha_1} - \alpha_2\beta_2\right]^2$$

Case 2: Think of X as being a conditional dist given θ.

Take the pdf to be $f(x|\theta) = f_\theta(x)$

Let the weight function be given by $p(\theta) \geq 0$. We treat this as a pdf for θ so

$$\int p(\theta)\,d\theta = 1$$

We get a new pdf

$$f(x) = \int_{-\infty}^{\infty} f_\theta(x)\,p(\theta)\,d\theta$$

thought of as marginal pdf of X.

ex] Let $X_\theta \sim Pois(\theta)$

Let the weight function be a pdf of θ
In particular, take $\theta \sim \Gamma(\alpha, \beta)$

For $x = 0, 1, 2, \ldots$ the pmf of the compound dist is

$$f(x) = \int_0^\infty f_\theta(x)\, p(\theta)\, d\theta$$

$$= \int_0^\infty \left[\frac{\theta^x e^{-\theta}}{x!}\right]\left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\theta^{\alpha-1} e^{-\frac{\theta}{\beta}}\right] d\theta$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha x!}\int_0^\infty \theta^{\alpha+x-1} e^{-\left(\theta+\frac{\theta}{\beta}\right)} d\theta$$

$$-\theta\left(\frac{1+\beta}{\beta}\right)$$

Let $u = \dfrac{\theta(1+\beta)}{\beta}$

$$\Rightarrow \frac{\beta}{1+\beta}\, du = d\theta$$

$$\Rightarrow \frac{\beta}{1+\beta}\, u = \theta$$

$\left(\frac{\beta}{1+\beta}\right)^{\alpha+x-1}$ 8

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha} x!} \int_0^{\infty} \left[\frac{\beta}{1+\beta} u\right]^{\alpha+x-1} e^{-u} \frac{\beta}{1+\beta}\, du$$

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha} x!} \int_0^{\infty} u^{\alpha+x-1} e^{-u} \left(\frac{\beta}{1+\beta}\right)^{\alpha+x} du$$

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha} x!} \frac{\beta^{\alpha+x}}{(1+\beta)^{\alpha+x}} \underbrace{\int_0^{\infty} u^{(\alpha+x)-1} e^{-u}\, du}$$

Gamma Function.

$$\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1} e^{-y}\, dy$$

$$= \frac{\beta^{x}}{\Gamma(\alpha)\, x!\, (1+\beta)^{\alpha+x}} \Gamma(\alpha+x)$$

for $x = 0, 1, 2, \ldots$