HW Sec 5.2 : 1 (use MGF technique)
2 (use CDF of Y from exercise 5.1.5)
7 (use MGF technique)
due Mon 12/4/17
This is the last collected HW.
All 3 problems will be graded for credit.

Sec 5.2 - Convergence In Distribution.

We know $\bar{X}_n \xrightarrow{P} \mu$ i.e.

$$\lim_{n\to\infty} \mathbb{P}\{|\bar{X}_n-\mu| > \varepsilon\} = 0$$

If we have a sequence X_n, what we care is when n is large. want to know the distribution of X_n, may or may not know the target.

ex) Suppose $X \sim N(0,1)$ (symmetric about 0)

Let $X_n = \begin{cases} X, & \text{if } n \text{ is odd} \\ -X, & \text{if } n \text{ is even.} \end{cases}$

No specific target for X_n as $n\to\infty$, but X_n has the same dist as $N(0,1)$.

def) Convergence in Distribution.

Let $\{X_n\}$ be a sequence of RVs.
Let X be a RV.

Let F_{X_n} be the CDF of X_n

Let F_X be the CDF of X.

Let $C(F_X)$ denote the set of all points where F_X is continuous.

We say $X_n \longrightarrow X$ i.d. (in distribution)

if $F_{X_n} \to F_X$ for all $x \in C(F_X)$

We write $X_n \xrightarrow{D} X$.

Remark) $X_n \xrightarrow{D} X$ only means F_{X_n} converges. It does NOT mean $X_n \xrightarrow{P} X$.

ex) $X_n = \begin{cases} X, & \text{if } n = \text{odd} \\ -X, & \text{if } n = \text{even}. \end{cases}$

$X \sim N(0,1)$

$$F_{X_n} = F_X \quad \text{so} \quad X_n \xrightarrow{D} X$$

But X_n doesn't get close to X,
so $X_n \overset{P}{\not\to} X$

Remark) Don't worry too much about $C(F_X)$.

In the case where X is a continuous RV. then
$$C(F_X) = \mathbb{R}$$

then conv. in. dist becomes

$$F_{X_n} \to F_X, \text{ for all } x \in \mathbb{R}$$

Remark) Conv. i.p. and i.d. falls under the idea of asymptotic theory.

The dist of X is the limiting/asymptotic dist. of the sequence $\{X_n\}$.

discrete

ex. suppose X_n has the ~~pdf~~ pmf

$$p_n(x) = \begin{cases} 1, & x = 2 + n^{-1} \\ 0, & \text{o.w.} \end{cases}$$

$$\lim_{n\to\infty} p_n(x) = 0 \quad \text{for all } x.$$

This may suggest that X_n, $n = 1, 2, 3, \ldots$ doesn't converge in dist.

However:
CDF for X_n is

$$F_n(x) = \begin{cases} 0, & x < 2 + n^{-1} \\ 1, & x \geq 2 + n^{-1} \end{cases}$$

Now

$$\lim_{n\to\infty} F_n(x) = \begin{cases} 0, & x \leq 2 \\ 1, & x > 2 \end{cases}$$

$$= F_X(x).$$

This is a cdf and $\lim_{n\to\infty} F_n(x) = F(x)$ for all $x \in C(F_X)$

then $X_n \xrightarrow{D} X$

Thm) Convergence i.p.

$\Rightarrow$ Convergence in dist

i.e. $X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{D} X$

Proof) Fix $\varepsilon > 0$

Let x be a continuity point of $F_X(x)$.

Assume $X_n \xrightarrow{P} X$.

$$F_{X_n}(x) = \mathbb{P}[X_n \leq x] \quad \text{def CDF}$$

$$= \mathbb{P}[X_n \leq x, X \leq x+\varepsilon]$$

$$+ \mathbb{P}[X_n \leq x, X > x+\varepsilon] \text{ ; LOTP}$$

ex) $\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c)$

$$\leq \mathbb{P}[X \leq x+\varepsilon] + \mathbb{P}[|X_n - X| > \varepsilon]$$

$$= F_X(x+\varepsilon) + \mathbb{P}[|X_n - X| > \varepsilon]$$

Also

$$\cancel{F_{X_n}} \; F_X(x-\varepsilon) = \mathbb{P}[X \leq x-\varepsilon]$$

$$= \mathbb{P}[X \leq x-\varepsilon, X_n \leq x]$$

$$+ \mathbb{P}[X \leq x-\varepsilon, X_n > x]$$

$$\le \mathbb{P}[X_n \le x] + \mathbb{P}[|X_n - X| > \varepsilon]$$

$$= F_{X_n}(x) + \mathbb{P}[|X_n - X| > \varepsilon].$$

$$F_X(x-\varepsilon) - \underbrace{\mathbb{P}[|X_n - X| > \varepsilon]}_{\to 0 \text{ as } n\to\infty}$$

$$\le F_{X_n}(x)$$

$$\le F_X(x+\varepsilon) + \underbrace{\mathbb{P}[|X_n - X| > \varepsilon]}_{\to 0 \text{ as } n\to\infty}$$

Take limit as $n \to \infty$

$$\left.\begin{aligned} F_X(x-\varepsilon) &\le \liminf_{n\to\infty} F_{X_n}(x) \\ &\le \limsup_{n\to\infty} F_{X_n}(x) \\ &\le F_X(x+\varepsilon) \end{aligned}\right\} \text{True for all } \varepsilon > 0.$$

Take the limit as $\varepsilon \downarrow 0$.
Use the fact that F is continuous at x.

So

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

$$\therefore X_n \xrightarrow{D} X$$

$\square$

Remark In general $X_n \xrightarrow{D} X \not\Rightarrow X_n \xrightarrow{P} X$.

But there is one case where it works.

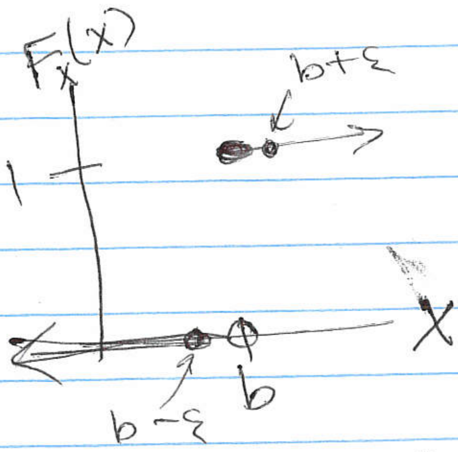
Thm If $X_n \xrightarrow{D} b$ (constant)

then $X_n \xrightarrow{P} b$

Proof Let $\varepsilon > 0$. Assume $X_n \xrightarrow{D} b$.

$$\lim_{n\to\infty} \mathbb{P}\{|X_n - b| \le \varepsilon\}$$

$$= \lim_{n\to\infty} \mathbb{P}\{X_n \le b+\varepsilon\}$$



$$- \lim_{n\to\infty} \mathbb{P}\{X_n \le b-\varepsilon\}$$

why: $X_n - b \le \varepsilon \Rightarrow X_n \le b+\varepsilon$
or $b - X_n \le \varepsilon \Rightarrow b-\varepsilon \le X_n$

$$= \lim_{n\to\infty} F_{X_n}(b+\varepsilon)$$

$$- \lim_{n\to\infty} F_{X_n}(b-\varepsilon)$$

$$= F_X(b+\varepsilon) - F_X(b-\varepsilon) = 1 - 0 = 1$$

Thm) If $X_n \xrightarrow{D} X$ and g is a continuous function, then

$$g(X_n) \xrightarrow{D} g(X)$$