+ 401- 11/29/17

For HW due last class:

3.7.1 - look @ my solutions

3.7.3 - remember there is a correction term for variances

5.1.3 - need to state as $n \to \infty$ so we know what limit we're taking

Sec 5.2

Recall: $X_n \xrightarrow{D} X$ if

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x)$$

for all $x \in C(F_X)$

Recall the example:

Suppose $X \sim N(0,1)$

Let $X_n = \begin{cases} X, & \text{if } n = \text{odd} \\ -X, & \text{if } n = \text{even}. \end{cases}$

$X_n \xrightarrow{D} X$ but $X_n \xrightarrow{P} X$

For example, if X is continuous then

$X_n \xrightarrow{D} X$ means that

$$
\begin{aligned}
F_{X_n}(b) - F_{X_n}(a) &= \mathbb{P}\{a < X_n \le b\} \\
&\to \mathbb{P}\{a < X \le b\} \\
&= F_X(b) - F_X(a)
\end{aligned}
$$

What if X is not continuous, say when $X = c$ (a constant).

ex) $X = 0$

What do we mean by $X_n \xrightarrow{D} 0$?

The CDF of X is $F_X(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 \le x. \end{cases}$

i.e.

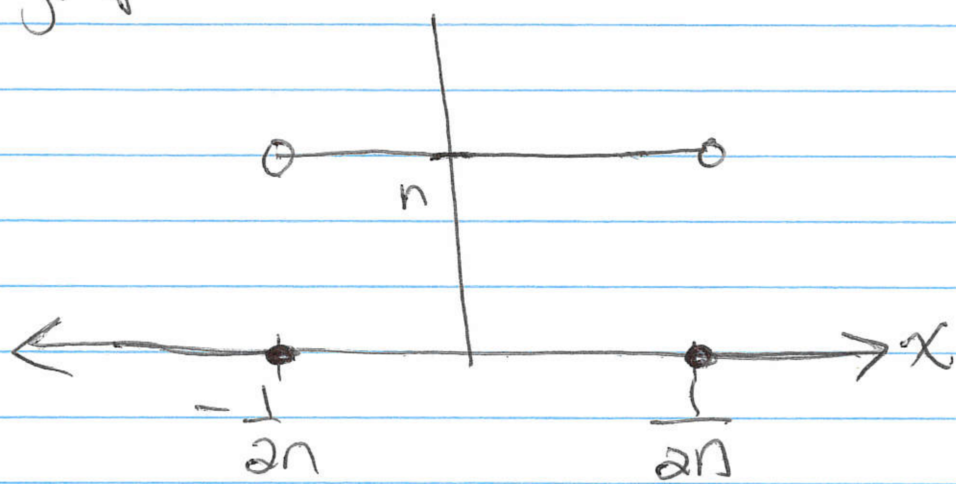
$$f_X(x) = \begin{cases} 1, & x = 0 \\ 0, & \text{o.w.} \end{cases}$$

This is called a degenerate distribution at 0

If X_n has pdf

$$f_{X_n}(x) = \begin{cases} n, & -\frac{1}{2n} < x < \frac{1}{2n} \\ 0, & \text{o.w.} \end{cases}$$

graph of pdf.



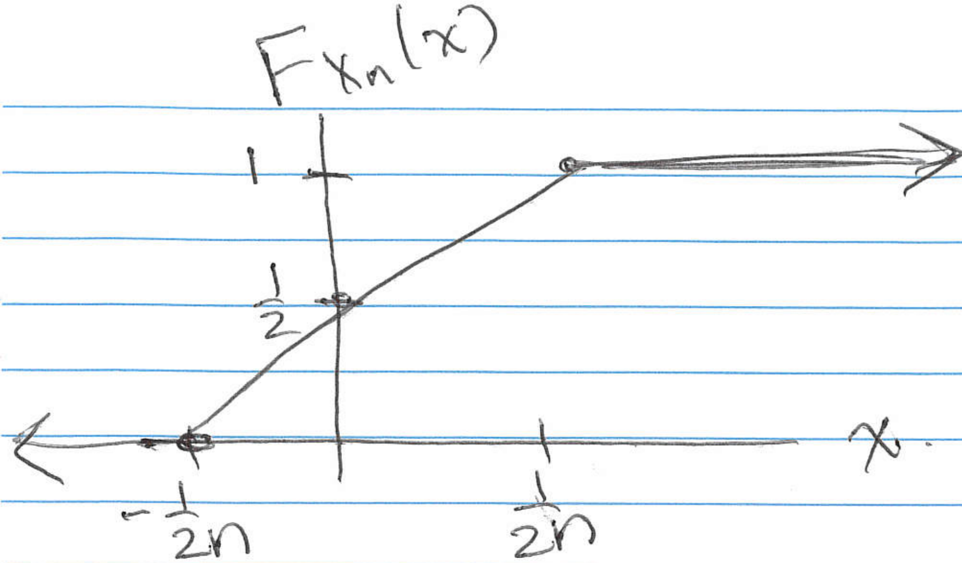
Find CDF of X_n.

$$P[X_n \le x] = \int_{-\frac{1}{2n}}^{x} n\,dt = nt \Big|_{-\frac{1}{2n}}^{x}$$

$$= nx - n\left(-\frac{1}{2n}\right) = nx + \frac{1}{2}$$

The CDF of X_n is

$$F_{X_n}(x) = \begin{cases} 0, & x < -\frac{1}{2n} \\ nx + \frac{1}{2}, & -\frac{1}{2n} \le x < \frac{1}{2n} \\ 1, & \frac{1}{2n} \le x \end{cases}$$



Note: For a degenerate dist, i.e. $X = c$

then $M_X(t) = e^{ct}$

ex) $M_X(t) = e^{2t}$

$X = 2$.

$$f_X(x) = \begin{cases} 1, & x = 2 \\ 0, & \text{o.w.} \end{cases}$$

Properties of Conv. in Dist

1. $X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{D} X$

2. If $X_n \xrightarrow{D} X$ and $g(x)$ is a continuous function then

$$g(X_n) \xrightarrow{D} g(X)$$

3 Warning:

If $X_n \xrightarrow{D} X$ and $Y_n \xrightarrow{D} Y$

it is NOT always the case that

$$X_n + Y_n \xrightarrow{D} X + Y$$

$$X_n Y_n \xrightarrow{D} XY$$

However, if $Y_n \xrightarrow{D} y$ (constant)

then

$$X_n Y_n \xrightarrow{D} yX \quad \text{and}$$

$$X_n + Y_n \xrightarrow{D} X + y$$

4 (Slutsky's Thm)

Let X_n, X, A_n, and B_n be RVs.

Let a, b be constants.

If $X_n \xrightarrow{D} X$, $A_n \xrightarrow{P} a$, $B_n \xrightarrow{P} b$

then $A_n + B_n X_n \xrightarrow{D} a + bX$

use MGF technique.

<u>Thm</u> Suppose X_n has MGF $M_n(t)$

Suppose X has MGF $M(t)$

If $\lim_{n\to\infty} M_n(t) = M(t)$

then $X_n \xrightarrow{D} X$.

<u>ex</u> Suppose $Y_n \sim Bin(n,p)$

Then $Y_n = Z_1 + \cdots + Z_n$

where $Z_i \sim Ber(p)$

Goal: Find the limiting dist of Binomial dist.

$$M_n(t) = \mathbb{E}[e^{tY_n}]$$

mgf for Y_n

$$= \mathbb{E}[e^{tZ_1}]\,\mathbb{E}[e^{tZ_2}]\cdots\mathbb{E}[e^{tZ_n}]$$

$$= ((1-p)+pe^t)^n$$

Aside: $\mu = np \Rightarrow p = \frac{\mu}{n}$

take μ = constant.

$$= \left(1 - \frac{\mu}{n} + \frac{\mu}{n} e^t\right)^n$$

$$= \left(1 + \frac{\mu}{n}(e^t - 1)\right)^n$$

$$= \left(1 + \frac{\mu(e^t-1)}{n}\right)^n$$

From Calc. it is known that

$$\lim_{n\to\infty} \left(1 + \frac{b}{n}\right)^{cn} = e^{bc}$$

For us $b = \mu(e^t - 1)$, $c = 1$

$$\longrightarrow e^{\mu(e^t-1)} \quad \text{as } n \to \infty$$

MGF for Poisson RV
with mean $= \mu$

We can use this approx when
n is large, (so p is small)

note: $$\lim_{n\to\infty}\left[1+\frac{b}{n}+\frac{\psi(n)}{n}\right]^{cn}$$

$$=\lim_{n\to\infty}\left[1+\frac{b}{n}\right]^{cn}$$

$$= e^{bc}$$

when b and c don't depend on n and $\lim_{n\to\infty}\psi(n)=0$

Thm] Delta Theorem / Method.

Suppose T_n is a statistic and as $n\to\infty$

$$\sqrt{n}\,(\underbrace{T_n-\theta}_{\text{we estimate } \theta \text{ with } T_n.}) \xrightarrow{D} N(0, v_\theta)$$

asymptotic variance depends on θ

Let $g(t)$ be a differentiable function at $t=\theta$ such that $g'(\theta)\neq 0$.

then

$$\sqrt{n}\left\{ g(T_n) - g(\theta) \right\} \xrightarrow{D} N(0, [g'(\theta)]^2 v_\theta)$$

ex) μ = pop mean.

$\bar{X}_n$ = sample mean.

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{D} N(0, \sigma^2)$$

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{D} N(0,1)$$