Stat 401 - 12/8/17

21 $X_1, \ldots, X_n \overset{iid}{\sim} \text{Unif}(0, \theta)$.

unknown.

Let $Y_n = \max\{X_1, \ldots, X_n\}$.

a) $F_{Y_n}(t) = \begin{cases} 1, & t > \theta \\ \left(\frac{t}{\theta}\right)^n, & 0 < t \le \theta \\ 0, & t \le 0. \end{cases}$

If $X_i \sim \text{Unif}(0, \theta)$.

$$f(x) = \frac{1}{\theta - 0} = \frac{1}{\theta}, \quad 0 < x < \theta$$

$\mathbb{P}[Y_n \le t] = \mathbb{P}[X_1 \le t, X_2 \le t, \ldots, X_n \le t]$

WLOG Let $Y_n = X_1$

largest of X_i's.

$\overset{indep}{=} \mathbb{P}[X_1 \le t] \cdot \ldots \cdot \mathbb{P}[X_n \le t]$

$= \left(\mathbb{P}[X_1 \le t]\right)^n$

what is $\mathbb{P}[X \le t]$

$$= \int_0^t \frac{1}{\theta}\, dt$$

$$= \frac{t}{\theta}\Big|_0^t = \frac{t}{\theta} - \frac{0}{\theta} = \frac{t}{\theta}$$

$$= \left(\mathbb{P}\left[X_1 \le t\right]\right)^n = \left(\frac{t}{\theta}\right)^n$$

for $0 < t < \theta$

b) Find the PDF for Y_n.

$$f_{Y_n}(t) = \begin{cases} \frac{n}{\theta^n} \cdot t^{n-1}, & 0<t<\theta \\ 0, & \text{o.w.} \end{cases}$$

c) Show Y_n is a biased estimator of θ.

Recall unbiased est means $\mathbb{E}[Y_n] = \theta$

if $\mathbb{E}[Y_n] \ne \theta$ then biased.

$E(X) = \int x f(x) dx$

$$\mathbb{E}[Y_n] = \int_0^\theta t \cdot \frac{n}{\theta^n} t^{n-1} \, dt$$

$$= \frac{n}{\theta^n} \int_0^\theta t^n \, dt$$

$$= \frac{n}{\theta^n} \frac{t^{n+1}}{n+1} \Big|_0^\theta$$

$$= \frac{n}{\theta^n} \frac{\theta^{n+1}}{n+1} \cancel{- \frac{n}{\theta^n} \cdot 0}$$

$$= \frac{n}{n+1} \theta \neq \theta$$

$\therefore$ Biased.

d) Show $\frac{n+1}{n} Y_n$ is unbiased for θ.

Show $\mathbb{E}\left[\frac{n+1}{n} Y_n\right] = \theta$.

$$\mathbb{E}\left[\frac{n+1}{n} Y_n\right] = \frac{n+1}{n} \mathbb{E}[Y_n]$$

$$= \frac{n+1}{n} \cdot \frac{n}{n+1} \theta = \theta \checkmark$$

e) Show $Y_n \xrightarrow{P} \theta$

Show $\lim_{n \to \infty} \mathbb{P}\left\{ |Y_n - \theta| > \varepsilon \right\} = 0$

$\mathbb{P}\left\{ |Y_n - \theta| > \varepsilon \right\}$

$= \mathbb{P}\left\{ \theta - Y_n > \varepsilon \right\}$

we have $0 < t < \theta$

ex) $0 < t < 2$

$Y_n = 1$

$|Y_n - \theta| = |1-2| = |-1|$

$= 1$

ex) If use A instead of Y.

$0 < a < \theta$

$$A = \begin{cases} \frac{n}{\theta^n} a^{n-1}, & 0 < a < \theta \\ 0, & o.w. \end{cases}$$

$$= \mathbb{P}\{-Y_n > \varepsilon - \theta\}$$

$$= \mathbb{P}\{Y_n < \theta - \varepsilon\}$$

$$= F_{Y_n}(\theta - \varepsilon)$$

$$= \left(\frac{\theta - \varepsilon}{\theta}\right)^n$$

$$= \left(1 - \frac{\varepsilon}{\theta}\right)^n$$

Since $\varepsilon > 0$ it is "small"

WLOG assume $0 < \varepsilon < \theta$

$$\Rightarrow 0 < \frac{\varepsilon}{\theta} < 1$$

$$\Rightarrow 1 - \frac{\varepsilon}{\theta} < 1$$

$$\lim_{n\to\infty} \left(1 - \frac{\varepsilon}{\theta}\right)^n = 0$$

$\therefore$ $\mathbb{P}\{|Y_n - \theta| > \varepsilon\} \longrightarrow 0$

as $n \to \infty$.

$\therefore$ $Y_n \xrightarrow{P} \theta$

f) Show $\frac{n+1}{n} Y_n \xrightarrow{P} \theta$.

$$0 \le \mathbb{P}\left\{ \left| \frac{n+1}{n} Y_n - \theta \right| > \varepsilon \right\}$$

$$= \mathbb{P}\left\{ \frac{n+1}{n} \left| Y_n - \frac{n}{n+1}\theta \right| > \varepsilon \right\}$$

$$= \mathbb{P}\left\{ \left| Y_n - \underbrace{\frac{n}{n+1}\theta}_{\mathbb{E}[Y_n]} \right| > \frac{n}{n+1}\varepsilon \right\}$$

Chebyshev's Ineq.

$$\mathbb{P}\left\{ \left| X - \underbrace{\mu}_{\mathbb{E}[X]} \right| > \varepsilon \right\} \le \frac{Var[X]}{\varepsilon^2}$$

$$\le \frac{Var[Y_n]}{\left(\frac{n}{n+1}\varepsilon\right)^2} = \frac{\left[\frac{n}{n+2} - \frac{n^2}{(n+1)^2}\right]\theta^2}{\frac{n^2}{(n+1)^2}\varepsilon^2}$$

$$\cdots \longrightarrow 0 \text{ as } n \to \infty.$$

23 Consider $Z_n = n(\theta - Y_n)$

Let $t \in (0, n\theta)$

Show $Z_n \xrightarrow{D} Exp(\theta) = Z$

One method:

Show $\lim_{n \to \infty} F_{Z_n}(t) = F_Z(t)$

$$
\begin{aligned}
\mathbb{P}[Z_n \le t] &= \mathbb{P}[n(\theta - Y_n) \le t] \\
&= \mathbb{P}\left[\theta - Y_n \le \frac{t}{n}\right] \\
&= \mathbb{P}\left[-Y_n \le \frac{t}{n} - \theta\right] \\
&= \mathbb{P}\left[Y_n \ge \theta - \frac{t}{n}\right] \\
&= 1 - \mathbb{P}\left(Y_n \le \theta - \frac{t}{n}\right) \\
&= 1 - F_{Y_n}\left(\theta - \frac{t}{n}\right) \\
&= 1 - \left(\frac{\theta - \frac{t}{n}}{\theta}\right)^n \\
&= 1 - \left(1 - \frac{t}{n\theta}\right)^n
\end{aligned}
$$

Monday 12 - 3 ; thurs 9 - 2 or 3

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$$= 1 - \left(1 + \frac{-t/\theta}{n}\right)^n$$

Recall

$$\lim_{n\to\infty} \left[1 + \frac{b}{n} + \frac{\psi(n)}{n}\right]^{cn}$$

$$= \lim_{n\to\infty} \left[1 + \frac{b}{n}\right]^{cn}$$

$$= e^{bc}$$

$b = -t/\theta \qquad c = 1.$

$$\lim_{n\to\infty} \mathbb{P}[Z_n \le t]$$

$$= \lim_{n\to\infty} 1 - \left(1 + \frac{-t/\theta}{n}\right)^n$$

$$= 1 - e^{-\frac{t}{\theta}} \leftarrow$$ CDF for exponential RV w/parameter θ.

$$\therefore Z_n \xrightarrow{D} Z = Exp(\theta)$$