Notes:

- THIS STUDY GUIDE COVERS SECTIONS 1.1–1.10
- You should also study all of your old homework assignments and in-class notes. Possible exam questions may come from those as well.
- REMINDERS: No cheat sheet. You may use a scientific, *but not graphing* calculator.

Section 1.1: Introduction

1. What is a sample space?

2. When we say that an experiment is random, what do we mean?

Section 1.2: Set Theory

3. Suppose $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Let $A = \{1, 4, 6, 8, 9\}$, $B = \{2, 3, 6, 7\}$, and $C = \{9, 10\}$. Find

   (a) $A \cap C$
   (b) $A \cup C$
   (c) $A \cap B$
   (d) $A \cup B$
   (e) $B \cup C$
   (f) $B \cap C$
   (g) A^C
   (h) B^C
   (i) $A^C \cap B$
   (j) $(A \cup B)^C$
   (k) $A^C \cup B'$
   (l) Are any of the events that you calculated above mutually exclusive? If so, which ones?

4. Which of the following are true for all events A, B, C, and which are not?

   (a) $\left(A^C \cup B^C\right) \cap C = (A \cap B)^C \cup C$
   (b) $A \cap (A \cap C)^C = \varnothing$
   (c) $\left(A^C \cup B^C\right) \cap (A \cap B)^C = \varnothing$
   (d) $\left(A^C \cup B^C\right) \subset (A \cap B)^C$
   (e) $B \subset \left(B \cap A^C\right) \cup A$

5. For every one-dimensional set C for which the integral exists, let $Q(C) = \int_C f(x)\,dx$, where $f(x) = 6x(1-x)$, $0 < x < 1$, zero elsewhere; otherwise, let $Q(C)$ be undefined. If $C_1 = \{x\colon \frac{1}{4} < x < \frac{3}{4}$, $C_2 = \{\frac{1}{2}\}$, and $C_3 = \{x\colon 0 < x < 10\}$, find $Q(C_1)$, $Q(C_2)$, and $Q(C_3)$.

Section 1.3: Probability Set Function

6. How many ways can you plant 5 different trees in a row?

7. How many ways can you choose 3 mystery books from 10 mystery books?

8. A president and a treasurer are chosen from a student club consisting of 50 people. How many different choices of officers are possible if
   (a) there are no restrictions?
   (b) A will serve only if he is president?

9. What does it mean for two events to be mutually exclusive?

10. Of 2500 freshmen in a certain school, 1000 are women; 1200 weigh over 140 pounds; of the women, 700 are taller than 5 ft 5 in; and of the men, 1300 are taller than 5 ft 5 in. A student is to be chosen at random from the freshman class. What is the probability that
    (a) a male student will be chosen?
    (b) a student weighing over 140 pounds will be chosen?
    (c) a student taller than 5 ft 5 in will be chosen?
    (d) a man shorter than 5 ft 5 in will be chosen?

11. Suppose A and B are independent events with $\mathbb{P}(A) = 0.6$ and $\mathbb{P}(B) = 0.3$ Find the following:
    (a) $\mathbb{P}(A \cup B)$
    (b) $\mathbb{P}(A' \cap B)$
    (c) $\mathbb{P}(A' \cup B')$
    (d) $\mathbb{P}(A \mid B)$
    (e) $\mathbb{P}(B' \mid A')$

12. Suppose A and B are mutually exclusive events with $\mathbb{P}(A) = 0.6$ and $\mathbb{P}(B) = 0.3$. Find the following:
    (a) $\mathbb{P}(A \cup B)$
    (b) $\mathbb{P}(A' \cap B)$
    (c) $\mathbb{P}(A' \cup B')$
    (d) $\mathbb{P}(A \mid B)$
    (e) $\mathbb{P}(B' \mid A')$

Section 1.4: Conditional Probability and Independence

13. Police often set up sobriety checkpoints - roadblocks where drivers are asked a few brief questions to allow the officer to judge whether or not the person may have been drinking. If the officer does not suspect a problem, drivers are released to go on their way. Otherwise, drivers are detained for a Breathalyzer test that will determine whether or not they will be arrested. The police say that based on the brief initial stop, trained officers can make the right decision 80% of the time. Suppose the police operate a sobriety checkpoint after 9:00 pm on a Saturday night, a time when national traffic safety experts suspect that about 12% of drivers have been drinking.

    Questions to answer:

    (a) You are stopped at the checkpoint and, of course, have not been drinking. What's the probability that you are detained for further testing?
    (b) What's the probability that any given driver will be detained?
    (c) What's the probability that a driver who is detained has actually been drinking?
    (d) What's the probability that a driver who was released had actually been drinking?

14. A company's records indicate that on any given day about 1% of their day-shift employees and 2% of the night-shift employees will miss work. Sixty percent of the employees work the day shift. What percent of employees are absent on any given day?

15. Given $\mathbb{P}(A) = 1/2$, $\mathbb{P}(B) = 1/3$, $\mathbb{P}(A \cap B) = 1/4$; find

    (a) $\mathbb{P}(A \cup B)$
    (b) $\mathbb{P}(A \mid B)$
    (c) $\mathbb{P}(B \mid A)$
    (d) $\mathbb{P}(A \cup B \mid B)$.

16. Eight tickets, numbered 111, 121, 122, 122, 211, 212, 212, 221 are placed in a brown hat and stirred. One is then to be drawn at random. Show that the events:

    A: "the first digit on the ticket drawn will be 1",
    B: "the second digit on the ticket drawn will be 1",
    C: "the third digit on the ticket drawn will be 1"

    are not pairwise independent (hence not mutually independent)l, although $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$.

Section 1.5: Random Variables

17. Suppose you have a random variable X. In the context in which we discussed random variables during class, why is $\mathscr{D}$ called a range?

18. Given the cdf

$$F(x) = \begin{cases} 0, & x < -2 \\ \frac{x+3}{5}, & -2 < x < 1 \\ 1, & 1 \leq x, \end{cases}$$

   (a) sketch the graph of $F(x)$
   (b) compute $\mathbb{P}\left(-\frac{3}{2} < X \leq \frac{1}{2}\right)$
   (c) $\mathbb{P}(X = 0)$
   (d) $\mathbb{P}(X = 1)$
   (e) $\mathbb{P}(X = -2)$.

19. Let $p_X(x)$ be the pmf of a random variable X. Find the cdf $F(x)$ of X and sketch its graph along with that of $p_X(x)$ if:

   (a) $p_X(x) = 1, \ \ x = 0$, zero elsewhere.
   (b) $p_X(x) = \frac{1}{3}, \ \ x = -1, 0, 1$, zero elsewhere.
   (c) $p_X(x) = x/15, \ \ x = 1, 2, 3, 4, 5$, zero elsewhere.

Section 1.6 / 1.8: Discrete Random Variables and Transformations and Expectations

20. The probability distribution of a random variable X is given by

$$p_X(x) = \begin{cases} \binom{3}{x} \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{3-x}, & x = 0, 1, 2, 3 \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Find the expected value of X.
   (b) Find the variance of X.
   (c) Find the CDF of X.

21. Let X have the pmf

$$p(x) = \left(\frac{1}{2}\right)^{|x|}, \quad x = -1, -2, -3, \ldots.$$

Find the pmf of $Y = X^4$.

22. Let X have a pmf $p(x) = \frac{1}{4}$, $x = 1, 2, 3, 4$, zero elsewhere.

   (a) Find the CDF of X.
   (b) Find $\mathbb{E}(X)$.
   (c) Find $\mathbb{E}(X^2)$.
   (d) Find $\mathbb{E}(-3X) + \mathbb{E}(4X^2)$.
   (e) Find $\mathbf{Var}(X)$.
   (f) Find the pmf of $Y = 2X + 1$.

Section 1.7 / 1.8: Continuous Random Variables and Transformations and Expectations

23. A continuous random variable X that can assume values between $x = 1$ and $x = 3$ has a density function given by $f_X(x) = 1/2$.

   (a) Find $F_X(x)$.
   (b) Use the CDF to evaluate $\mathbb{P}(2 < X < 2.5)$.
   (c) Use the PDF to evaluate $\mathbb{P}(2 < X < 2.5)$.
   (d) Did you get the same answer in the previous parts?
   (e) Use the CDF to evaluate $\mathbb{P}(X = 2)$.
   (f) Use the CDF to evaluate $\mathbb{P}(X = 3)$.

24. On a laboratory assignment, if the equipment is working, the density function of the observed outcome, X, is

$$f_X(x) = \begin{cases} 2(1-x), & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Calculate $\mathbb{P}(X \leq 1/3)$.
   (b) What is the probability that X will exceed 0.5?
   (c) Given that $X \geq 0.5$, what is the probability that X will be less than 0.75?

25. Suppose X has the PDF

$$f_X(x) = \begin{cases} cx^3, 0 < x < 2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the constant c such that $f_X(x)$ is a valid PDF.

(b) Find the probability that $X \leq 1$ if we already know that $X \geq 0.5$.

26. The density function of a certain continuous random variable X is proportional to $x(1-x)$ for $0 < x < 1$, and is 0 for other values of x.

(a) Find a value of c so that $f(x) = cx(1-x)$ for $0 < x < 1$ is a valid pdf.

(b) Find the CDF of X.

(c) Find $\mathbb{P}(X < 1/2)$.

(d) Find $\mathbb{E}(X)$.

(e) Find $\mathbb{E}(X^2)$.

(f) Find $\mathbb{E}(5X) - \mathbb{E}(3X^2)$.

(g) Find $\mathbf{Var}(X)$.

(h) Find the CDF and PDF of a random variable Y, if $Y = X^2$.

(i) Find the PDF of a random variable Y, if $Y = -2\log X$.

Section 1.9: Some Special Expectations

27. For the following probability distributions [(a) – (c)], do the following:

i) Find the MGFs of the following probability distributions. Be sure to state if there are any restrictions on t.

ii) Find the mean and variance, if they exist. Do this using the MGF.

iii) Find the mean and variance using the definition for mean and variance.

Distributions:

(a) $p(x) = p^x\,(1-p)^{1-x}\,,\;\; x = 0, 1 \;\; and 0 < p < 1.$

(b) $p(x) = \dfrac{\lambda^x e^{-\lambda}}{x!},\;\; x = 0, 1, 2, \ldots$ and $\lambda > 0$.

(c) $f(x) = \theta e^{-\theta x},\;\; x > 0$ and $\theta > 0$.

Section 1.10: Important Inequalities

28. Let X be a random variable such that $\mathbb{P}(X \leq 0) = 0$ and let $\mu = \mathbb{E}(X)$ exist. Find an upper bound for $\mathbb{P}(X \geq 5\mu)$.

29. Computers from a particular company are found to last on average for three years without any hardware malfunction, with a standard deviation of two months. At least what percent of the computers last between 31 months and 41 months?

30. Let X have the PDF

$$f(x) = \begin{cases} \frac{1}{10}, & 0 < x < 10 \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Find the mean and variance of X.

   (b) Calculate an upper bound for $\mathbb{P}\left(|X - 5| \geq 3\right)$.

   (c) Calculate $\mathbb{P}\left(|X - 5| \geq 4\right)$.

Solutions

Section 1.1

1. What is a sample space?

   Solution:

   A sample space is a collection of all possible outcomes. Sometimes sample spaces are discrete (countable) or continuous (intervals of $\mathbb{R}$).

2. When we say that an experiment is random, what do we mean?

   Solution:

   We have an experiment (or some test) that we want to perform. We know what the possible outcomes are, but we do not know which of these outcomes will actually occur.

Section 1.2

3. Suppose $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Let $A = \{1, 4, 6, 8, 9\}$, $B = \{2, 3, 6, 7\}$, and $C = \{9, 10\}$. Find

   Solution:

   (a) $A \cap C = \{9\}$
   (b) $A \cup C = \{1, 4, 6, 8, 9, 10\}$
   (c) $A \cap B = \{6\}$
   (d) $A \cup B = \{1, 2, 3, 4, 6, 7, 8, 9\}$
   (e) $B \cup C = \{2, 3, 6, 7, 9, 10\}$
   (f) $B \cap C = \varnothing$
   (g) $A' = \{2, 3, 5, 7, 10\}$
   (h) $B' = \{1, 4, 5, 8, 9, 10\}$
   (i) $A' \cap B = \{2, 3, 7\}$
   (j) $(A \cup B)' = (A' \cap B') = \{5, 10\}$
   (k) $A' \cup B' = \{1, 2, 3, 4, 5, 7, 8, 9, 10\}$
   (l) Are any of the events that you calculated above mutually exclusive? If so, which ones?
   B and C are mutually exclusive because $B \cap C = \varnothing$.

4. Which of the following are true for all events A, B, C, and which are not?

   (a) $\left(A^C \cup B^C\right) \cap C = (A \cap B)^C \cup C$

   Solution:

   By DeMorgan's Laws:

   $$\left(A^C \cup B^C\right) \cap C = (A \cap B)^C \cap C \neq (A \cap B)^C \cup C.$$

   This statement is not true for all events A, B, C.

(b) $A \cap (A \cap C)^C = \varnothing$
Solution:

$$\begin{aligned} A \cap (A \cap C)^C &= A \cap \left(A^C \cup C^C\right) \\ &= \left(A \cap A^C\right) \cup \left(A \cap C^C\right) \\ &= \varnothing \cup \left(A \cap C^C\right) \\ &= A \cap C^C. \end{aligned}$$

$A \cap C^C$ will only equal $\varnothing$ if there is no overlap between these two events. However, it is possible that there may be an overlap so this statement is not necessarily true for all events A, B, C.

(c) $\left(A^C \cup B^C\right) \cap (A \cap B)^C = \varnothing$
Solution:

$$\left(A^C \cup B^C\right) \cap (A \cap B)^C = \left(A^C \cup B^C\right) \cap \left(A^C \cup B^C\right) = A^C \cup B^C = (A \cap B)^C.$$

$A^C \cup B^C$ will not be equal to $\varnothing$ unless A^C and B^C are both empty. The original statement is not necessarily true.

(d) $\left(A^C \cup B^C\right) \subset (A \cap B)^C$
Solution:

$$A^C \cup B^C = (A \cap B)^C$$

Since the LHS and the RHS, then $LHS \subset RHS$. This statement is true always.

(e) $B \subset \left(B \cap A^C\right) \cup A$
Solution:

$$\begin{aligned} \left(B \cap A^C\right) \cup A &= A \cup \left(B \cap A^C\right) \\ &= (A \cup B) \cap \left(A \cup A^C\right) \\ &= (A \cup B) \cap \mathscr{C} \\ &= A \cup B. \end{aligned}$$

Because $B \subset A \cup B$ is always true, the initial statement is true for all events.

5. For every one-dimensional set C for which the integral exists, let $Q(C) = \int_C f(x)\,dx$, where $f(x) = 6x(1-x)$, $0 < x < 1$, zero elsewhere; otherwise, let $Q(C)$ be undefined. If $C_1 = \{x\colon \frac{1}{4} < x < \frac{3}{4}$, $C_2 = \{\frac{1}{2}\}$, and $C_3 = \{x\colon 0 < x < 10\}$, find $Q(C_1)$, $Q(C_2)$, and $Q(C_3)$.

Solution:

$$Q(C_1) = \int_{1/4}^{3/4} 6x(1-x)\,dx = \int_{1/4}^{3/4} \left(6x - 6x^2\right)\,dx = 6\cdot\frac{x^2}{2}\bigg|_{1/4}^{3/4} - 6\cdot\frac{x^3}{3}\bigg|_{1/4}^{3/4}$$

$$= 6\left(\frac{(3/4)^2}{2} - \frac{(1/4)^2}{2}\right) - 6\left(\frac{(3/4)^3}{3} - \frac{(1/4)^3}{3}\right) = 6\left(\frac{1}{4}\right) - 6\left(\frac{13}{96}\right) = \boxed{\frac{11}{16}};$$

$$Q(C_2) = \int_{1/2}^{1/2} 6x(1-x)\,dx = 0;$$

$$Q(C_3) = \int_0^{10} f(x)\,dx = \int_0^1 6x(1-x)\,dx + \int_1^{10} 0\,dx = \int_0^1 \left(6x - 6x^2\right)\,dx + 0$$

$$= 6\cdot\frac{x^2}{2}\bigg|_0^1 - 6\cdot\frac{x^3}{3}\bigg|_0^1 = 6\left(\frac{1^2}{2} - 0\right) - 6\left(\frac{1^3}{3} - 0\right) = 6\left(\frac{1}{2}\right) - 6\left(\frac{1}{3}\right) = 3 - 2 = \boxed{1}.$$

Section 1.3

6. How many ways can you plant 5 different trees in a row?

Solution:

Use permutations. $\boxed{5! = 120}$.

7. How many ways can you choose 3 mystery books from 10 mystery books?

Solution:

Use combinations because you do not care the order in which the books are chosen.

$${}_{10}C_3 = \binom{10}{3} = \frac{10!}{3!(10-3)!} = \frac{10!}{3!7!} = \boxed{120}.$$

8. A president and a treasurer are chosen from a student club consisting of 50 people. How many different choices of officers are possible if

   (a) there are no restrictions?

   Solution:

   Order matters. Use permutations.

   $$\boxed{{}_{50}P_2 = \frac{50!}{(50-2)!} = 2450}.$$

   (b) A will serve only if he is president?

   Solution:

   Person A will serve only if he is president. We have 2 options: 1) A is president; 2) A is not president. In Option 1: there are $n_1 = 1$ ways of selecting the president, and $n_2 = 49$ ways of selecting the treasurer, for a total of $n_1 n_2 = (1)(49) = 49$ ways of selecting the president and treasurer. In Option 2: there are $n_1 = 49$ ways of selecting the president, and $n_2 = 48$ ways of selecting the treasurer (remember that A does not want to be treasurer), for at total of $n_1 n_2 = (49)(48) = 2352$ ways of selecting the president and treasurer. In total, there are $\boxed{49 + 2352 = 2401}$ ways.

9. What does it mean for two events to be mutually exclusive?

   Solution:

   It means that they are disjoint, or there is no overlap between events. If A and B are the events, then if they are mutually exclusive, $A \cap B = \varnothing$.

10. Of 2500 freshmen in a certain school, 1000 are women; 1200 weigh over 140 pounds; of the women, 700 are taller than 5 ft 5 in; and of the men, 1300 are taller than 5 ft 5 in. A student is to be chosen at random from the freshman class. What is the probability that

   (a) a male student will be chosen?
   Solution:
$$\mathbb{P}(Male) = \frac{2500 - 1000}{2500} = \frac{1500}{2500} = \boxed{\frac{3}{5}}.$$

   (b) a student weighing over 140 pounds will be chosen?
   Solution:
$$\mathbb{P}(\text{Weighs over 140 lbs}) = \frac{1200}{2500} = \boxed{\frac{12}{25}}.$$

   (c) a student taller than 5 ft 5 in will be chosen?
   Solution:
$$\mathbb{P}\left(\text{Taller than 5 ft 5 in}\right) = \frac{700 + 1300}{2500} = \frac{2000}{2500} = \boxed{\frac{4}{5}}.$$

   (d) a man shorter than 5 ft 5 in will be chosen?
   Solution:
$$\mathbb{P}\left(\text{man shorter than 5 ft 5 in}\right) = \frac{(2500 - 1000) - 1300}{2500} = \frac{200}{2500} = \boxed{\frac{2}{25}}.$$

11. Suppose A and B are independent events with $\mathbb{P}(A) = 0.6$ and $\mathbb{P}(B) = 0.3$ Find the following:

   Solution:

   Remember that independent events mean that $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

   (a) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = 0.6 + 0.3 - \mathbb{P}(A)\mathbb{P}(B) = 0.9 - (0.6)(0.3) = 0.72$
   (b) $\mathbb{P}(A' \cap B) = \mathbb{P}(A')\mathbb{P}(B) = [1 - \mathbb{P}(A)]\,\mathbb{P}(B) = [1 - 0.6]\,(0.3) = (0.4)(0.3) = 0.12$
   (c) $\mathbb{P}(A' \cup B') = \mathbb{P}(A') + \mathbb{P}(B') - \mathbb{P}(A' \cap B') = 0.4 + 0.7 - \mathbb{P}(A')\mathbb{P}(B') = 1.1 - (0.4)(0.7) = 0.82$
   (d) $\mathbb{P}(A \mid B) = \mathbb{P}(A) = 0.6$
   (e) $\mathbb{P}(B' \mid A') = \mathbb{P}(B') = 0.7$

12. Suppose A and B are mutually exclusive events with $\mathbb{P}(A) = 0.6$ and $\mathbb{P}(B) = 0.3$. Find the following:

Solution:

Remember that mutually exclusive means that $\mathbb{P}(A \cap B) = 0$.

(a) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = 0.6 + 0.3 - 0 = 0.9$

(b) $\mathbb{P}(A' \cap B) = \mathbb{P}(B) = 0.3$.
This is because if you draw a Venn diagram, the circles for A and B do not overlap. A' refers to the area outside of the circle A, which includes the circle B. The overlap between A' and B would then be circle B.

(c) $\mathbb{P}(A' \cup B') = \mathbb{P}(A \cap B)' = 1 - \mathbb{P}(A \cap B) = 1 - 0 = 1$

(d) $\mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \dfrac{0}{0.3} = 0$

(e) $\mathbb{P}(B' \mid A') = \dfrac{\mathbb{P}(B' \cap A')}{\mathbb{P}(A')} = \dfrac{\mathbb{P}(B \cup A)'}{0.4} = \dfrac{1 - \mathbb{P}(A \cup B)}{0.4} = \dfrac{1 - 0.9}{0.4} = \dfrac{0.1}{0.4} = 0.25$

Section 1.4

13. Police often set up sobriety checkpoints - roadblocks where drivers are asked a few brief questions to allow the officer to judge whether or not the person may have been drinking. If the officer does not suspect a problem, drivers are released to go on their way. Otherwise, drivers are detained for a Breathalyzer test that will determine whether or not they will be arrested. The police say that based on the brief initial stop, trained officers can make the right decision 80% of the time. Suppose the police operate a sobriety checkpoint after 9:00 pm on a Saturday night, a time when national traffic safety experts suspect that about 12% of drivers have been drinking.

Questions to answer:

(a) You are stopped at the checkpoint and, of course, have not been drinking. What's the probability that you are detained for further testing?

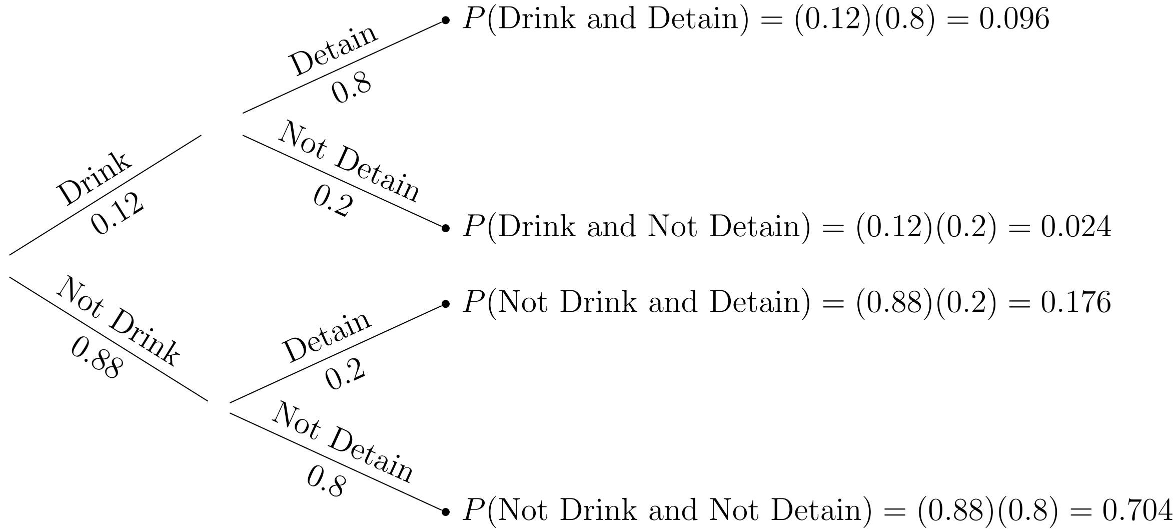
(b) What's the probability that any given driver will be detained?

(c) What's the probability that a driver who is detained has actually been drinking?

(d) What's the probability that a driver who was released had actually been drinking?

Solution:

Before we answer any questions, it may be useful to create a tree diagram.



Questions to answer:

(a) You are stopped at the checkpoint and, of course, have not been drinking. What's the probability that you are detained for further testing?
Solution:

$$\mathbb{P}(\text{Detain} \mid \text{Not Drink}) = \boxed{0.2}.$$

(b) What's the probability that any given driver will be detained?
Solution:

$$\mathbb{P}(\text{Detain}) = \mathbb{P}(\text{Detain and Drink}) + \mathbb{P}(\text{Detain and Not Drink}) = 0.096 + 0.176 = \boxed{0.272}.$$

(c) What's the probability that a driver who is detained has actually been drinking?
Solution:

$$\mathbb{P}(\text{Drink} \mid \text{Detain}) = \frac{\mathbb{P}(\text{Drink and Detain})}{\mathbb{P}(\text{Detain})} = \frac{0.096}{0.272} = \boxed{0.353}.$$

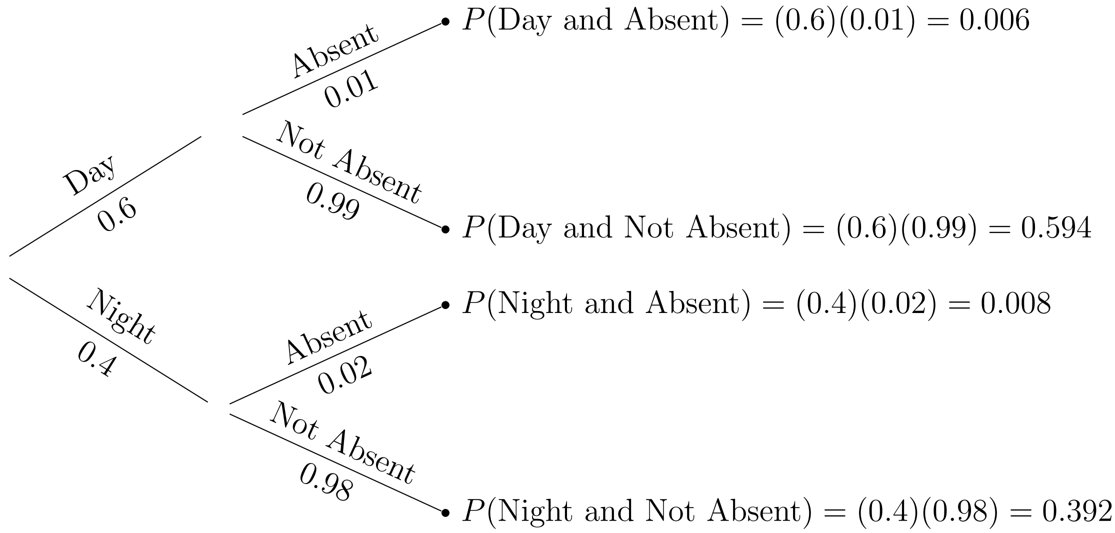
(d) What's the probability that a driver who was released had actually been drinking?
Solution:

$$\begin{aligned}\mathbb{P}(\text{Drink} \mid \text{Not Detain}) &= \frac{\mathbb{P}(\text{Not Detain} \mid \text{Drink})\mathbb{P}(\text{Drink})}{\mathbb{P}(\text{Not Detain} \mid \text{Drink})\mathbb{P}(\text{Drink}) + \mathbb{P}(\text{Not Detain} \mid \text{Not Drink})\mathbb{P}(\text{Not Drink})} \\ &= \frac{(0.2)(0.12)}{(0.2)(0.12) + (0.8)(0.88)} = \boxed{0.033}.\end{aligned}$$

14. A company's records indicate that on any given day about 1% of their day-shift employees and 2% of the night-shift employees will miss work. Sixty percent of the employees work the day shift. What percent of employees are absent on any given day?

Solution:

Before we answer any questions, it may be useful to create a tree diagram.



What percent of employees are absent on any given day?
Need to calculate P(Absent). This is the denominator of Bayes Rule.

$$\begin{aligned}\mathbb{P}(\text{Absent}) &= \mathbb{P}(\text{Absent} \mid \text{Day})\mathbb{P}(\text{Day}) + \mathbb{P}(\text{Absent} \mid \text{Night})\mathbb{P}(\text{Night}) \\ &= (0.01)(0.6) + (0.02)(0.4) = 0.014 = \boxed{1.4\%}.\end{aligned}$$

15. Given $\mathbb{P}(A) = 1/2$, $\mathbb{P}(B) = 1/3$, $\mathbb{P}(A \cap B) = 1/4$; find

(a) $\mathbb{P}(A \cup B)$
Solution:

$$\begin{aligned}\mathbb{P}(A \cup B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \\ &= \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \boxed{\frac{7}{12}}.\end{aligned}$$

(b) $\mathbb{P}(A \mid B)$
Solution:

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{1/4}{1/3} = \boxed{\frac{3}{4}}.$$

(c) $\mathbb{P}(B \mid A)$
Solution:

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{1/4}{1/2} = \boxed{\frac{1}{2}}.$$

(d) $\mathbb{P}(A \cup B \mid B)$.
Solution:

$$\begin{aligned}\mathbb{P}\left(A \cup B \mid B\right) &= \frac{\mathbb{P}\left([A \cup B] \cap B\right)}{\mathbb{P}(B)} \\ &= \frac{\mathbb{P}\left([A \cap B] \cup [B \cap B]\right)}{1/3} \\ &= \frac{\mathbb{P}\left([A \cap B] \cup B\right)}{1/3} \\ &= \frac{\mathbb{P}(A \cap B) + \mathbb{P}(B) - \mathbb{P}(A \cap B \cap B)}{1/3} \\ &= \frac{1/4 + 1/3 - \mathbb{P}(A \cap B)}{1/3} \\ &= \frac{1/4 + 1/3 - 1/4}{1/3} = \frac{1/3}{1/3} = \boxed{1}.\end{aligned}$$

16. Eight tickets, numbered 111, 121, 122, 122, 211, 212, 212, 221 are placed in a brown hat and stirred. One is then to be drawn at random. Show that the events:

A: "the first digit on the ticket drawn will be 1",
B: "the second digit on the ticket drawn will be 1",
C: "the third digit on the ticket drawn will be 1"

are not pairwise independent (hence not mutually independent)l, although $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$.

Solution:

We have

$$\begin{aligned} \mathbb{P}(A) &= \frac{4}{8} = \frac{1}{2}; & \mathbb{P}(A \cap B) &= \frac{1}{8}; & \mathbb{P}(A \cap B \cap C) &= \frac{1}{8} \\ \mathbb{P}(B) &= \frac{4}{8} = \frac{1}{2}; & \mathbb{P}(A \cap C) &= \frac{2}{8} = \frac{1}{4}; \\ \mathbb{P}(C) &= \frac{4}{8} = \frac{1}{2}; & \mathbb{P}(B \cap C) &= \frac{2}{8} = \frac{1}{4}. \end{aligned}$$

Pairwise independent says if we take any two events, then $\mathbb{P}(C_i \cap C_j) = \mathbb{P}(C_i) \cdot \mathbb{P}(C_j), \ \forall i \neq j$.

$$\frac{1}{8} = \mathbb{P}(A \cap B) \neq \mathbb{P}(A) \cdot \mathbb{P}(B) = \frac{1}{4}.$$

Since these two pairs do not work, then we do not have pairwise independence because it must work for all pairs of events. We also do not have mutual independence because all groups of events must have the independence property. Note that

$$\frac{1}{8} = \mathbb{P}(A \cap B \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C) = \frac{1}{8}.$$

Section 1.5

17. Suppose you have a random variable X. In the context in which we discussed random variables during class, why is $\mathscr{D}$ called a range?

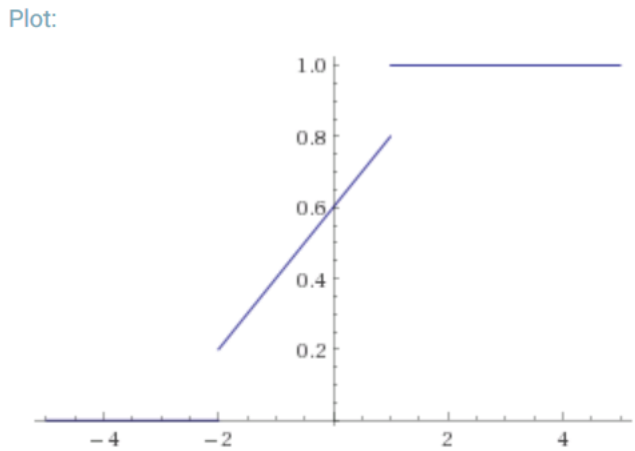
Solution:

We start out with our original sample space $\mathscr{C}$. X is a function that maps values in $\mathscr{C}$ to values in $\mathscr{D}$. In this context, $\mathscr{D}$ is the range of the function.

18. Given the cdf

$$F(x) = \begin{cases} 0, & x < -2 \\ \frac{x+3}{5}, & -2 \le x < 1 \\ 1, & 1 \le x, \end{cases}$$

(a) sketch the graph of $F(x)$
Solution:



(b) compute $\mathbb{P}\left(-\frac{3}{2} < X \le \frac{1}{2}\right)$
Solution:

$$\mathbb{P}\left(-\frac{3}{2} < X \le \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F\left(-\frac{3}{2}\right) = \frac{0.5+3}{5} - \frac{-1.5+3}{5} = \boxed{\frac{2}{5}}.$$

(c) $\mathbb{P}(X = 0)$
Solution:

$$\mathbb{P}(X = 0) = F(0) - F(0-) = \frac{0+3}{5} - \frac{0+3}{5} = \boxed{0}.$$

(d) $\mathbb{P}(X = 1)$
Solution:

$$\mathbb{P}(X = 1) = F(1) - F(1-) = 1 - \frac{1+3}{5} = 1 - \frac{4}{5} = \boxed{\frac{1}{5}}.$$

(e) $\mathbb{P}(X = -2)$.
Solution:

$$\mathbb{P}(X = -2) = F(-2) - F(-2-) = \frac{-2+3}{5} - 0 = \boxed{\frac{1}{5}}.$$

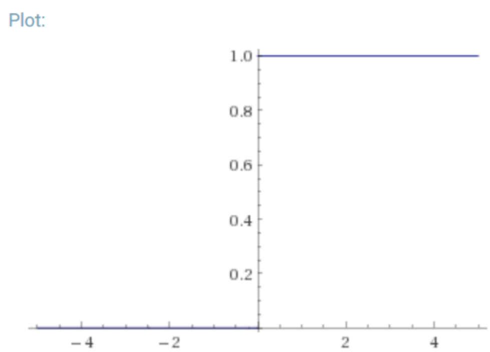
19. Let $p_X(x)$ be the pmf of a random variable X. Find the cdf $F(x)$ of X and sketch its graph along with that of $p_X(x)$ if:

(a) $p_X(x) = 1, \quad x = 0$, zero elsewhere.

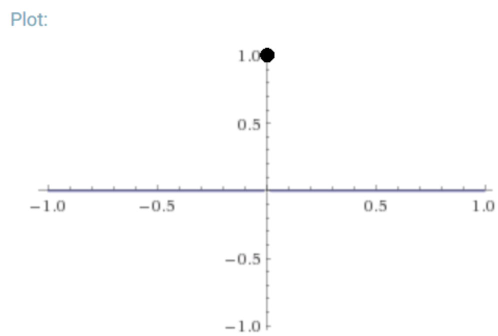
Solution:
The CDF is

$$F(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0. \end{cases}$$

(a) CDF



(b) PMF

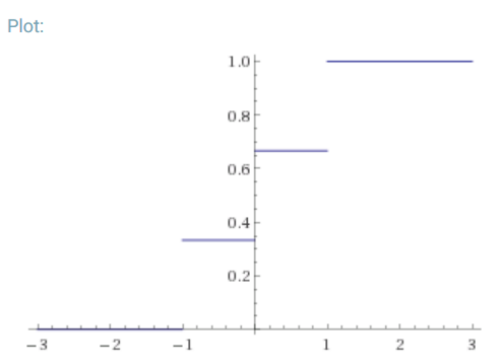


(b) $p_X(x) = \frac{1}{3}, \quad x = -1, 0, 1$, zero elsewhere.

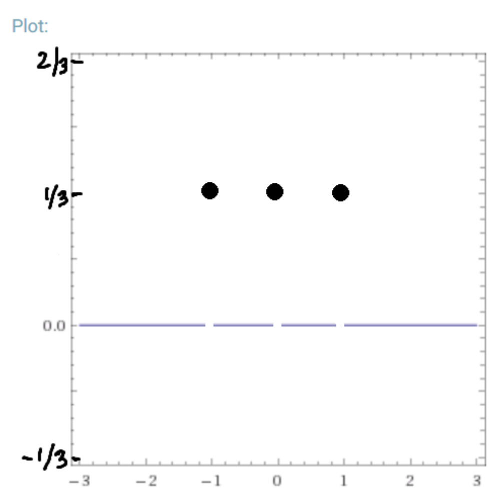
Solution:
The CDF is

$$F(x) = \begin{cases} 0, & x < -1 \\ \frac{1}{3}, & -1 \leq x < 0 \\ \frac{2}{3}, & 0 \leq x < 1 \\ 1, & 1 \leq x. \end{cases}$$

(a) CDF



(b) PMF



(c) $p_X(x) = x/15, \quad x = 1, 2, 3, 4, 5$, zero elsewhere.

Solution:

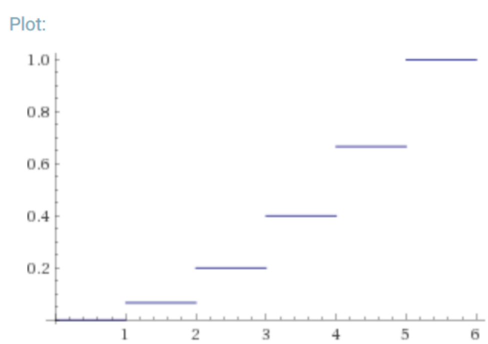
The PDF is

x	1	2	3	4	5
$p_X(x)$	1/15	2/15	3/15	4/15	5/15

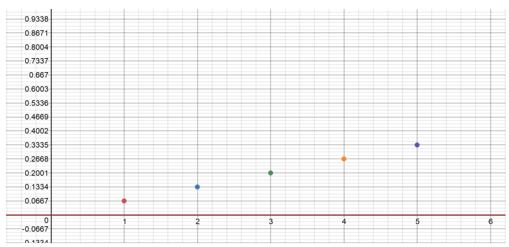
The CDF is

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{15}, & 1 \leq x < 2 \\ \frac{1}{5}, & 2 \leq x < 3 \\ \frac{2}{5}, & 3 \leq x < 4 \\ \frac{2}{3}, & 4 \leq x < 5 \\ 1, & 5 \leq x. \end{cases}$$

(a) CDF



(b) PMF



Section 1.6 / 1.8

20. The probability distribution of a random variable X is given by

$$p_X(x) = \begin{cases} \binom{3}{x}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{3-x}, & x = 0, 1, 2, 3 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the expected value of X.

Solution:

$$\begin{aligned} \mathbb{E}[X] &= \sum_{x=0}^{3} x \cdot \binom{3}{x}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{3-x} \\ &= 0 + 1 \cdot \binom{3}{1}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^2 + 2 \cdot \binom{3}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right) + 3 \cdot \binom{3}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^0 \\ &= \frac{27}{64} + \frac{9}{32} + \frac{3}{64} = \boxed{\frac{3}{4}}. \end{aligned}$$

(b) Find the variance of X.
Solution:

$$\begin{aligned}
\mathbb{E}[X^2] &= \sum_{x=0}^{3} x^2 \cdot \binom{3}{x}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{3-x} \\
&= 0 + 1^2 \cdot \binom{3}{1}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^2 + 2^2 \cdot \binom{3}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right) + 3^2 \cdot \binom{3}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^0 \\
&= \frac{27}{64} + \frac{9}{16} + \frac{9}{64} = \frac{9}{8}; \\
\mathbf{Var}[X] &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{9}{8} - \left(\frac{3}{4}\right)^2 = \boxed{\frac{9}{16}}.
\end{aligned}$$

(c) Find the CDF of X.
Solution: The PMF of X is

x	0	1	2	3
$p_X(x)$	27/64	27/64	9/64	1/64

The CDF of X is

$$\boxed{F(x) = \begin{cases} 0, & x < 0 \\ 27/64, & 0 \leq x < 1 \\ 54/64, & 1 \leq x < 2 \\ 63/64, & 2 \leq x < 3 \\ 1, & 3 \leq x \end{cases}}$$

21. Let X have the pmf

$$p(x) = \left(\frac{1}{2}\right)^{|x|}, \quad x = -1, -2, -3, \ldots.$$

Find the pmf of $Y = X^4$.

Solution:

Normally $y = x^4$ would not define a one-to-one transformation. However, here it does because there are no positive values of x in $\mathscr{D}_X = \{x \colon x = -1, -2, -3, \ldots\}$. The transformation $y = g(x) = x^4$ maps $\mathscr{D}_X = \{x = -1, -2, -3, \ldots\}$ onto

$$\mathscr{D}_Y = \{y \colon y = (-1)^4, (-2)^4, (-3)^4, \ldots\} = \{y \colon y = 1, 16, 81, \ldots\}.$$

Note that $x = g^{-1}(y) = -\sqrt[4]{y}$.

$$p_Y(y) = p_X\left(-\sqrt[4]{y}\right) = \left(\frac{1}{2}\right)^{\left|-\sqrt[4]{y}\right|} = \boxed{\left(\frac{1}{2}\right)^{\sqrt[4]{y}}, \quad y = 1, 16, 81, \ldots}$$

22. Let X have a pmf $p(x) = \frac{1}{4}$, $x = 1, 2, 3, 4$, zero elsewhere.

(a) Find the CDF of X.
Solution:

$$\boxed{F(x) = \begin{cases} 0, & x < 1 \\ 1/4, & 1 \le x < 2 \\ 1/2, & 2 \le x < 3 \\ 3/4, & 3 \le x < 4 \\ 1, & 4 \le x \end{cases}}$$

(b) Find $\mathbb{E}(X)$.
Solution:

$$\mathbb{E}[X] = \sum_{x=1}^{4} x \cdot \frac{1}{4} = \left(\frac{1}{4}\right)(1 + 2 + 3 + 4) = \boxed{\frac{5}{2}}.$$

(c) Find $\mathbb{E}(X^2)$.
Solution:

$$\mathbb{E}[X^2] = \sum_{x=1}^{4} x^2 \cdot \frac{1}{4} = \left(\frac{1}{4}\right)\left(1^2 + 2^2 + 3^2 + 4^2\right) = \boxed{\frac{15}{2}}.$$

(d) Find $\mathbb{E}(-3X) + \mathbb{E}(4X^2)$.
Solution:

$$\mathbb{E}[-3X] + \mathbb{E}[4X^2] = -3\mathbb{E}[X] + 4\mathbb{E}[X^2] = -3\left(\frac{5}{2}\right) + 4\left(\frac{15}{2}\right) = \boxed{\frac{45}{2}}.$$

(e) Find $\textbf{Var}(X)$.
Solution:

$$\textbf{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{15}{2} - \left(\frac{5}{2}\right)^2 = \boxed{\frac{5}{4}}.$$

(f) Find the pmf of $Y = 2X + 1$.

Solution:
Because $y = g(x) = 2x + 1$, we have a one-to-one transformation. In this case, $x = \frac{y-1}{2} = g^{-1}(y)$. The space of Y is $\mathscr{D}_Y = \{3, 5, 7, 9\}$ The pmf of Y is

$$p_Y(y) = p_X\left(\frac{y-1}{2}\right) = \boxed{\frac{1}{4}, \text{ for } y = 3, 5, 7, 9}.$$

Section 1.7 / 1.8

23. A continuous random variable X that can assume values between $x = 1$ and $x = 3$ has a density function given by $f_X(x) = 1/2$.

(a) Find $F_X(x)$.
Solution:

$$\int_1^x \frac{1}{2}\, dt = \frac{1}{2}t\Big|_1^x = \frac{1}{2}x - \frac{1}{2}(1) = \frac{1}{2}(x-1).$$

The CDF of X is

$$\boxed{F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{2}(x-1), & 1 \le x < 3 \\ 1, & 3 \le x. \end{cases}}$$

(b) Use the CDF to evaluate $\mathbb{P}(2 < X < 2.5)$.

Solution:
Because the CDF is continuous,

$$\mathbb{P}(2 < X < 2.5) = \mathbb{P}(2 < X \le 2.5) = F(2.5) - F(2) = \frac{3}{4} - \frac{1}{2} = \boxed{\frac{1}{4}}.$$

(c) Use the PDF to evaluate $\mathbb{P}(2 < X < 2.5)$.
Solution:

$$\mathbb{P}(2 < X < 2.5) = \int_2^{2.5} \frac{1}{2}\, dx = \frac{1}{2}x\Big|_2^{2.5} = \frac{1}{2}(2.5 - 2) = \frac{1}{2}\left(\frac{1}{2}\right) = \boxed{\frac{1}{4}}.$$

(d) Did you get the same answer in the previous parts?

Solution:
Yes, the answers in parts (b) and (c) were the same. This is what was supposed to happen.

(e) Use the CDF to evaluate $\mathbb{P}(X = 2)$.
Solution:

$$\mathbb{P}(X = 2) = F_X(2) - F_X(2-) = \frac{1}{2}(2-1) - \frac{1}{2}(2-1) = \boxed{0}.$$

(f) Use the CDF to evaluate $\mathbb{P}(X = 3)$.
Solution:

$$\mathbb{P}(X = 3) = F_X(3) - F_X(3-) = 1 - \frac{1}{2}(3-1) = \boxed{0}.$$

24. On a laboratory assignment, if the equipment is working, the density function of the observed outcome, X, is

$$f_X(x) = \begin{cases} 2(1-x), & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Calculate $\mathbb{P}(X \le 1/3)$.
Solution:

$$\begin{aligned}\mathbb{P}(X \le 1/3) &= \int_0^{1/3} 2(1-x)\,dx = \int_0^{1/3} (2-2x)\,dx = 2x - 2\cdot\frac{x^2}{2}\Big|_0^{1/3} \\ &= 2x - x^2\Big|_0^{1/3} = 2(1/3) - (1/3)^2 - \big(2(0) - 0^2\big) = \frac{2}{3} - \frac{1}{9} = \boxed{\frac{5}{9}}.\end{aligned}$$

(b) What is the probability that X will exceed 0.5?
Solution:

$$\begin{aligned}\mathbb{P}(X > 0.5) &= \int_{0.5}^{1} 2(1-x)\,dx = \int_{0.5}^{1} (2-2x)\,dx = 2x - x^2\Big|_{0.5}^{1} \\ &= 2(1) - (1)^2 - \big(2(0.5) - (0.5)^2\big) = 2 - 1 - (1 - 1/4) = \boxed{\frac{1}{4}}.\end{aligned}$$

(c) Given that $X \ge 0.5$, what is the probability that X will be less than 0.75?

Solution:
This is an example of conditional probability. I first find $\mathbb{P}(0.5 \le X < 0.75)$.

$$\begin{aligned}\mathbb{P}(0.5 \le X < 0.75) &= \int_{0.5}^{0.75} (2-2x)\,dx = 2x - x^2\Big|_{0.5}^{0.75} \\ &= 2(0.75) - (0.75)^2 - \big(2(0.5) - (0.5)^2\big) \\ &= \frac{15}{16} - \left(\frac{3}{4}\right) = \frac{3}{16}.\end{aligned}$$

Now I calculate the conditional probability.

$$\mathbb{P}(X < 0.75 \mid X \ge 0.5) = \frac{\mathbb{P}(X < 0.75 \cap X \ge 0.5)}{\mathbb{P}(X \ge 0.5)} = \frac{\mathbb{P}(0.5 \le X < 0.75)}{\mathbb{P}(X < 0.5)} = \frac{3/16}{1/4} = \boxed{\frac{3}{4}}.$$

25. Suppose X has the PDF

$$f_X(x) = \begin{cases} cx^3, 0 < x < 2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the constant c such that $f_X(x)$ is a valid PDF.
Solution:

$$1 = \int_0^2 cx^3\,dx = c\frac{x^4}{4}\Big|_0^2 = c\left(\frac{2^4}{4} - \frac{0^4}{4}\right) = 4c \Rightarrow \boxed{c = \frac{1}{4}}$$

(b) Find the probability that $X \le 1$ if we already know that $X \ge 0.5$.
Solution:

$$\begin{aligned}\mathbb{P}\left(X \le 1 \mid X \ge 0.5\right) &= \frac{\mathbb{P}(X \le 1 \cap X \ge 0.5)}{\mathbb{P}(X \ge 0.5)} = \frac{\mathbb{P}(0.5 \le X \le 1)}{\mathbb{P}(X \ge 0.5)} \\ &= \frac{\int_{0.5}^1 \frac{1}{4}x^3\,dx}{\int_{0.5}^2 \frac{1}{4}x^3\,dx} = \frac{\int_{0.5}^1 x^3\,dx}{\int_{0.5}^2 x^3\,dx} = \frac{\frac{x^4}{4}\Big|_{0.5}^1}{\frac{x^4}{4}\Big|_{0.5}^2} \\ &= \frac{\frac{1}{4}\left(1^4 - 0.5^4\right)}{\frac{1}{4}\left(2^4 - 0.5^4\right)} = \boxed{\frac{1}{17}}.\end{aligned}$$

26. The density function of a certain continuous random variable X is proportional to $x(1-x)$ for $0 < x < 1$, and is 0 for other values of x.

(a) Find a value of c so that $f(x) = cx(1-x)$ for $0 < x < 1$ is a valid pdf.
Solution:

$$\begin{aligned}1 &= \int_0^1 cx(1-x)\,dx = \int_0^1 (cx - cx^2)\,dx = c\frac{x^2}{2} - c\frac{x^3}{3}\Big|_0^1 = c\left(\frac{1^2}{2} - 0\right) - c\left(\frac{1^3}{3} - 0\right) \\ &= \frac{1}{6}c \Rightarrow \boxed{c = 6}.\end{aligned}$$

(b) Find the CDF of X.
Solution:

$$\int_0^x (6t - 6t^2)\,dt = 6\frac{t^2}{2} - 6\frac{t^3}{3}\Big|_0^x = 3(x^2 - 0) - 2(x^3 - 0) = 3x^2 - 2x^3 = x^2(3 - 2x)$$

The CDF is

$$\boxed{F(x) = \begin{cases} 0, & x < 0 \\ x^2(3-2x), & 0 \le x < 1 \\ 1, & 1 \le x \end{cases}}$$

(c) Find $\mathbb{P}(X < 1/2)$.
Solution:

$$\mathbb{P}(X < 1/2) = F(1/2) - F(0) = \left(\frac{1}{2}\right)^2\left(3 - 2\cdot\frac{1}{2}\right) - 0 = \boxed{\frac{1}{2}}.$$

(d) Find $\mathbb{E}(X)$.
Solution:

$$\mathbb{E}[X] = \int_0^1 x \cdot 6x(1-x)\,dx = \int_0^1 (6x^2 - 6x^3)\,dx = 6 \cdot \frac{x^3}{3} - 6 \cdot \frac{x^4}{4}\Big|_0^1 = 2x^3 - \frac{3}{2}x^4\Big|_0^1$$
$$= 2(1)^3 - \frac{3}{2}(1)^4 - (0-0) = 2 - \frac{3}{2} = \boxed{\frac{1}{2}}.$$

(e) Find $\mathbb{E}(X^2)$.
Solution:

$$\mathbb{E}[X^2] = \int_0^1 x^2 \cdot 6x(1-x)\,dx = \int_0^1 (6x^3 - 6x^4)\,dx$$
$$= 6 \cdot \frac{x^4}{4} - 6 \cdot \frac{x^5}{5}\Big|_0^1 = \frac{3}{2}(1)^4 - \frac{6}{5}(1)^5 - (0-0) = \boxed{\frac{3}{10}}.$$

(f) Find $\mathbb{E}(5X) - \mathbb{E}(3X^2)$.
Solution:

$$\mathbb{E}[5X] - \mathbb{E}[3X^2] = 5\mathbb{E}[X] - 3\mathbb{E}[X^2] = 5(1/2) - 3(3/10) = \boxed{\frac{8}{5}}.$$

(g) Find $\mathbf{Var}(X)$.
Solution:

$$\mathbf{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{3}{10} - \left(\frac{1}{2}\right)^2 = \boxed{\frac{1}{20}}.$$

(h) Find the CDF and PDF of a random variable Y, if $Y = X^2$.

Solution:
In this case, $0 < y < 1$. We can write the CDF of Y as

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \le y) = \mathbb{P}(X^2 \le y) = \mathbb{P}(X \le \sqrt{y});\ \text{since } 0 < x < 1 \\ &= F_X(\sqrt{y}) = \sqrt{y}^2\,(3 - 2\sqrt{y}) \\ &= \boxed{y\,(3 - 2\sqrt{y})\,,\ \ 0 \le y < 1}. \end{aligned}$$

The PDF of Y is

$$\boxed{f_Y(y) = \begin{cases} 3 - 3\sqrt{y}, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}}$$

(i) Find the PDF of a random variable Y, if $Y = -2\log X$.

Solution:
Since $0 < x < 1$, this means $0 < y < \infty$. Note that the transformation $g(x) = -2\log x$ is a one-to-one transformation. We can find that $x = g^{-1}(y) = e^{-y/2}$. The Jacobian of the transformation is

$$J = \frac{d}{dy}e^{-y/2} - \frac{1}{2}e^{-y/2}.$$

Note that

$$f_X\left(e^{-y/2}\right)|J| = 6e^{-y/2}\left(1 - e^{-y/2}\right)\left|-\frac{1}{2}e^{-y/2}\right| = 3e^{-y}\left(1 - e^{-y/2}\right).$$

The PDF of Y is

$$\boxed{f_Y(y) = \begin{cases} 3e^{-y}\left(1 - e^{-y/2}\right), & 0 < y < \infty \\ 0, & \text{otherwise.} \end{cases}}$$

Section 1.9

27. For the following probability distributions [(a) – (c)], do the following:

i) Find the MGFs of the following probability distributions. Be sure to state if there are any restrictions on t.

ii) Find the mean and variance, if they exist. Do this using the MGF.

iii) Find the mean and variance using the definition for mean and variance.

Distributions:

(a) $p(x) = p^x(1-p)^{1-x}, \quad x = 0, 1 \ \mathit{and} 0 < p < 1$

Solution:

i) MGF:

$$M(t) = \mathbb{E}\left[e^{tX}\right] = \sum_{x=0}^{1} e^{tx}p^x(1-p)^{1-x} = e^t p^1(1-p)^0 + e^0p^0(1-p)^1 = pe^t + 1 - p$$
$$= \boxed{1 - p + pe^t, \quad -\infty < t < \infty}.$$

ii) Find the mean and variance using the MGF.

$$\mathbb{E}[X] = M'(0) = pe^t\big|_{t=0} = pe^0 = \boxed{p};$$
$$\mathbf{Var}[X] = M''(0) - [M'(0)]^2 = pe^t\big|_{t=0} - p^2 = pe^0 - p^2 = p - p^2 = \boxed{p(1-p)}.$$

iii) Find the mean and variance using the pmf.

$$\mathbb{E}[X] = \sum_{x=0}^{1} x \cdot p^x(1-p)^{1-x} = 0 + 1 \cdot p^1(1-p)^{1-1} = \boxed{p}$$
$$\mathbb{E}[X^2] = \sum_{x=0}^{1} x^2 \cdot p^x(1-p)^{1-x} = 0 + 1^2 \cdot p^1(1-p)^{1-1} = p$$
$$\mathbf{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = p - p^2 = \boxed{p(1-p)}.$$

(b) $p(x) = \dfrac{\lambda^x e^{-\lambda}}{x!}$, $x = 0, 1, 2, \ldots$ and $\lambda > 0$.

Solution:

i) MGF:

$$M(t) = \mathbb{E}\left[e^{tX}\right] = \sum_{x=0}^{\infty} e^{tx} \cdot \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t\lambda)^x}{x!} = e^{-\lambda} e^{\lambda e^t} = e^{-\lambda + \lambda e^t}$$
$$= \boxed{e^{\lambda(e^t - 1)}, \quad -\infty < t < \infty}.$$

ii) Find the mean and variance using the MGF.

$$\mathbb{E}[X] = M'(0) = \lambda e^{t + \lambda e^t - \lambda}\Big|_{t=0} = \lambda e^{0 + \lambda e^0 - \lambda} = \lambda e^{\lambda - \lambda} = \boxed{\lambda}$$
$$\mathbf{Var}[X] = M''(0) - [M'(0)]^2$$
$$= \lambda e^{\lambda\left(e^t - 1\right) + t}\left(\lambda e^t + 1\right)\Big|_{t=0} - (\lambda)^2$$
$$= \lambda e^{\lambda\left(e^0 - 1\right) + 0}\left(\lambda e^0 + 1\right) - \lambda^2$$
$$= \lambda e^{\lambda(0) + 0}(\lambda + 1) - \lambda^2$$
$$= \lambda(\lambda + 1) - \lambda^2 = \boxed{\lambda}.$$

iii) Find the mean and variance using the pmf.
We find the value of the sum first.

$$\sum_{x=0}^{\infty} \frac{e^{-\mu}\mu^x}{x!} = e^{-\mu} \sum_{x=0}^{\infty} \frac{\mu^x}{x!}$$
$$= e^{-\mu} e^{\mu}; \text{ by the power series expansion for } e^y.$$
$$= 1.$$

Find the expected value.

$$\mathbb{E}[X] = \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda}(\lambda^x)}{x!}$$
$$= \sum_{x=1}^{\infty} x \cdot \frac{e^{-\lambda}(\lambda^x)}{x!}; \quad \text{When } x = 0\text{, the value of the sum is } 0.$$
$$= \sum_{x=1}^{\infty} \frac{e^{-\lambda}(\lambda^x)}{(x-1)!}; \qquad \frac{x}{x!} = \frac{1}{(x-1)!}$$
$$= \sum_{x=1}^{\infty} \frac{e^{-\lambda}(\lambda^{x-1})(\lambda)}{(x-1)!}; \qquad \lambda^x = (\lambda^{x-1})(\lambda)$$
$$= \lambda \sum_{x=1}^{\infty} \frac{e^{-\lambda}(\lambda^{x-1})}{(x-1)!}; \qquad \text{Pull the } \lambda \text{ out of the sum}$$

$$= \lambda \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^{y}}{y!} \qquad \text{Let } y = x - 1$$

$$= \lambda(1); \qquad \text{The sum is equal to 1 by what we showed before.}$$

$$= \boxed{\lambda}$$

$$\begin{aligned}
\mathbb{E}[X^2] &= \sum_{x=0}^{\infty} x^2 \cdot \frac{\lambda^x e^{-\lambda}}{x!} \\
&= \sum_{x=1}^{\infty} x^2 \cdot \frac{\lambda^x e^{-\lambda}}{x!}; \text{ When } x = 0\text{, the value of the sum is } 0 \\
&= \sum_{x=1}^{\infty} x \cdot \frac{\lambda^x e^{-\lambda}}{(x-1)!}; \qquad \frac{x^2}{x!} = \frac{x}{(x-1)!} \\
&= \sum_{x=1}^{\infty} \frac{x e^{-\lambda}\lambda^{x-1}\lambda}{(x-1)!} \\
&= \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{(x-1+1)\lambda^{x-1}}{(x-1)!} \\
&= \lambda e^{-\lambda} \left[\sum_{x=1}^{\infty} \frac{(x-1)\lambda^{x-1}}{(x-1)!} + \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} \right] \\
&= \lambda e^{-\lambda} \left[\lambda \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} \right]; \text{ In the first summation, when } x = 1, \sum = 0 \\
&= \lambda e^{-\lambda} \left[\lambda \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} + \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \right]; \text{ Let } i = x - 2 \text{ and } j = x - 1 \\
&= \lambda \left[\lambda \sum_{i=0}^{\infty} \frac{\lambda^i e^{-\lambda}}{i!} + \sum_{j=0}^{\infty} \frac{\lambda^j e^{-\lambda}}{j!} \right] \\
&= \lambda \left[\lambda(1) + 1\right] \\
&= \lambda^2 + \lambda
\end{aligned}$$

The variance is:

$$\mathbf{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \lambda^2 + \lambda - (\lambda)^2 = \boxed{\lambda}.$$

(c) $f(x) = \theta e^{-\theta x}$, $x > 0$ and $\theta > 0$.

i) MGF:

$$M(t) = \mathbb{E}\left[e^{tX}\right] = \int_0^\infty e^{tx}\theta e^{-\theta x}\,dx = \int_0^\infty \theta e^{-x(\theta - t)}\,dx$$
$$= \theta \cdot \frac{-1}{\theta - t} e^{-x(\theta-t)}\Big|_0^\infty$$
$$= \frac{\theta}{t-\theta}(0-1)\,;\ \text{where } t<\theta \text{ (or else } e^{-x(\theta-t)} \text{ would have a positive exponent)}$$
$$= \boxed{\frac{\theta}{\theta - t} = \frac{1}{1 - t/\theta},\quad t < \theta}.$$

ii) Find the mean and variance using the MGF.

$$\mathbb{E}[X] = M'(0) = \frac{\theta}{(\theta-t)^2}\Big|_{t=0} = \frac{\theta}{\theta^2} = \boxed{\frac{1}{\theta}};$$
$$\mathbf{Var}[X] = M''(0) - [M'(0)] = \frac{2\theta}{(\theta-t)^3}\Big|_{t=0} - \left(\frac{1}{\theta}\right)^2 = \frac{2\theta}{\theta^3} - \frac{1}{\theta^2} = \frac{2}{\theta^2} - \frac{1}{\theta^2} = \boxed{\frac{1}{\theta^2}}.$$

iii) Find the mean and variance using the pdf.

$$\mathbb{E}[X] = \int_0^\infty x\cdot\theta e^{-\theta x}\,dx = x\cdot\theta\cdot\frac{-1}{\theta}e^{-\theta x} - 1\cdot\theta\cdot\frac{1}{\theta^2}e^{-\theta x}\Big|_0^\infty$$
$$= 0 - 0 - \left(0 - \frac{\theta}{\theta^2}e^0\right) = \boxed{\frac{1}{\theta}}$$
$$\mathbb{E}[X^2] = \int_0^\infty x^2\cdot\theta e^{-\theta x}\,dx = \theta x^2\cdot\frac{-1}{\theta}e^{-\theta x} - 2\theta x\cdot\frac{1}{\theta^2}e^{-\theta x} + 2\theta\cdot\frac{-1}{\theta^3}e^{-\theta x}\Big|_0^\infty$$
$$= 0 - 0 - 0 - \left(0 - 0 - \frac{2}{\theta^2}e^0\right) = \frac{2}{\theta^2}$$
$$\mathbf{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{2}{\theta^2} - \left(\frac{1}{\theta}\right)^2 = \frac{2}{\theta^2} - \frac{1}{\theta^2} = \boxed{\frac{1}{\theta^2}}.$$

Section 1.10

28. Let X be a random variable such that $\mathbb{P}(X \leq 0) = 0$ and let $\mu = \mathbb{E}(X)$ exist. Find an upper bound for $\mathbb{P}(X \geq 5\mu)$.

Solution:

Because $\mathbb{P}(X \leq 0) = 0$, this means that $\mathbb{P}(X > 0) = 1$, so $X/\mu = u(X)$ is a nonnegative function (all values of x are nonnegative, and probabilities are nonnegative so μ is nonnegative)s. We also have

$$\mathbb{E}\left[\frac{X}{\mu}\right] = \frac{1}{\mu}\mathbb{E}[X] = 1,$$

so $\mathbb{E}[u(X)]$ exists. Let $c = 5$, so that c is a positive constant. We can use Markov's Inequality.

$$\mathbb{P}\left(X \geq 5\mu\right) = \mathbb{P}\left(\frac{X}{\mu} \geq 5\right) \leq \frac{\mathbb{E}\left[X/\mu\right]}{5} = \boxed{\frac{1}{5}}.$$

29. Computers from a particular company are found to last on average for three years without any hardware malfunction, with a standard deviation of two months. At least what percent of the computers last between 31 months and 41 months?

Solution:

We want to find $\mathbb{P}(31 < X < 41)$. Note that 3 years is the same as 36 months. Use Chebyshev's Inequality.

$$\begin{aligned}\mathbb{P}(31 < X < 41) &= \mathbb{P}\left(|X - 36| < 5\right) = \mathbb{P}\left(|X - 36| < \frac{5}{2} \cdot 2\right) = 1 - \mathbb{P}\left(|X - 36| \geq \frac{5}{2} \cdot 2\right) \\ &\geq 1 - \frac{1}{(5/2)^2} = 1 - \frac{4}{25} = \frac{21}{25} = 0.84.\end{aligned}$$

At least 84% of the computers last between 31 months and 41 months.

30. Let X have the PDF

$$f(x) = \begin{cases} \frac{1}{10}, & 0 < x < 10 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the mean and variance of X.
Solution:

$$\mathbb{E}[X] = \int_0^{10} x \cdot \frac{1}{10}\, dx = \frac{1}{10} \cdot \frac{x^2}{2}\bigg|_0^{10} = \frac{1}{20} \cdot 100 = \boxed{5};$$

$$\mathbb{E}[X^2] = \int_0^{10} x^2 \cdot \frac{1}{10}\, dx = \frac{1}{10} \cdot \frac{x^3}{3}\bigg|_0^{10} = \frac{1}{30} \cdot 1000 = \frac{100}{3};$$

$$\mathbf{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{100}{3} - (5)^2 = \boxed{\frac{25}{3}};$$

$$\mathbf{SD}[X] = \sqrt{\frac{25}{3}} = \frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{3}.$$

(b) Calculate an upper bound for $\mathbb{P}\left(|X - 5| \geq 3\right)$.

Solution:
Use Chebyshev's Inequality.

$$\mathbb{P}\left(|X-5| \geq 3\right) = \mathbb{P}\left(|X-5| \geq \frac{3\sqrt{3}}{5} \cdot \frac{5}{\sqrt{3}}\right) \leq \frac{1}{\left(\frac{3\sqrt{3}}{5}\right)^2} = \frac{1}{\frac{9(3)}{25}} = \boxed{\frac{25}{27}}.$$

(c) Calculate $\mathbb{P}\left(|X - 5| \geq 4\right)$.

Solution:
We do not use Chebyshev's Inequality here because we want to find the exact probability, not the upper bound.

$$\begin{aligned}\mathbb{P}\left(|X-5| \geq 4\right) &= \mathbb{P}(X-5 \geq 4) + \mathbb{P}(X-5 \leq -4) = \mathbb{P}(X \geq 9) + \mathbb{P}(X \leq 1) \\ &= \int_9^{10} \frac{1}{10}\, dx + \int_0^1 \frac{1}{10}\, dx = \frac{1}{10}x\bigg|_9^{10} + \frac{1}{10}x\bigg|_0^1 \\ &= \frac{1}{10}(10-9) + \frac{1}{10}(1-0) = \boxed{\frac{1}{5}}.\end{aligned}$$