Notes:

- THIS STUDY GUIDE COVERS SECTIONS 2.1–2.8; 3.1, 3.2
- You should also study all of your old homework assignments and in-class notes. Possible exam questions may come from those as well.
- REMINDERS: No cheat sheet. You may use a scientific, *but not graphing* calculator.

Section 2.1

1. Suppose you are given the following joint distribution for X_1 and X_2:

$$p_{1,2}(x_1, x_2) = \begin{cases} \frac{x_1+x_2}{30}, & x_1 = 0, 1, 2, 3; \;\; x_2 = 0, 1, 2, \\ 0, & \text{otherwise,} \end{cases}$$

   (a) Find $\mathsf{P}[X_1 \leq 1, X_2 > 0]$
   (b) Find $\mathsf{P}[X_1 > X_2]$.
   (c) Find $F_1(x_1)$, the CDF of X_1.
   (d) Make a table listing the marginal distribution of X_1.
   (e) Find $\mathbf{E}(X_1 X_2)$.
   (f) Find $\mathbf{E}(X_1)$.

2. Let X_1 and X_2 be random variables. Their joint distribution, $f(x_1, x_2)$, is given by

$$f(x_1, x_2) = \begin{cases} 10x_1x_2^2, & 0 < x_1 < x_2 < 1, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Find $\mathsf{P}\left[X_1 < 0.25, 0.5 < X_2 < 1\right]$.
   (b) Find $F_2(x_2)$, the CDF of X_2.
   (c) Find the marginal distribution of X_2.
   (d) Find $\mathbf{E}(X_1 X_2)$.
   (e) Find the marginal distribution for X_1.
   (f) Find $\mathbf{E}(X_1)$.
   (g) Find $\mathbf{E}(-5X_1)$.

Section 2.2 & 2.7

Note: You should be able to extend any of these types of problems to multiple random variables.

3. Let X_1 and X_2 be two random variables with joint probability distribution

$$p(x_1, x_2) = \begin{cases} (1-p_1)^{x_1-1} p_1 (1-p_2)^{x_2-1} p_2, & x_1 = 1, 2, \ldots;\ x_2 = 1, 2, \ldots;\ 0 < p_1, p_2 < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find the distribution of $Y_1 = X_1 + X_2$.

4. Suppose that X is a continuous random variable such that it has the pdf

$$f(x) = \begin{cases} \frac{1}{6}, & -2 \le x \le 4 \\ 0, & \text{otherwise.} \end{cases}$$

Define $Y = X^2$.

(a) Find the CDF of X. (b) Find the CDF of Y. (c) Find the PDF of Y.

5. Let X_1 and X_2 be two continuous random variables with joint probability distribution

$$f(x_1, x_2) = \begin{cases} \frac{1}{\pi}, & 0 < x_1^2 + x_2^2 < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Define $Y_1 = X_1^2 + X_2^2$ and $Y_2 = \frac{X_1^2}{X_1^2 + X_2^2}$. Find the joint pdf of Y_1 and Y_2.

Section 2.3

6. Let X_1 and X_2 be continuous random variables with joint probability distribution

$$f(x_1, x_2) = \begin{cases} \frac{5}{16} x_1 x_2^2, & 0 < x_1 < x_2 < 2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $\mathbf{E}[X_1 X_2]$.
(b) Find the marginal distribution of X_2.
(c) Find the conditional distribution of X_1, given $X_2 = x_2$.
(d) Find $\mathsf{P}\left[0 < X_1 < 1 \mid X_2 = \frac{3}{2}\right]$.
(e) Find $\mathsf{P}[0 < X_1 < 1]$.
(f) Find $\mathbf{E}[X_1]$.
(g) Find $\mathbf{V}[X_1]$.
(h) Find the distribution of $Y = \mathbf{E}[X_1 \mid X_2]$.
(i) Find $\mathbf{E}[Y]$.
(j) Find $\mathbf{V}[Y]$. How does this value compare to $\mathbf{V}[X_1]$?

Section 2.4

7. Let X and Y have the joint pmf described as follows:

(x,y)	$(0,1)$	$(0,3)$	$(0,5)$	$(1,1)$	$(1,3)$	$(1,5)$	$(2,1)$	$(2,5)$
$p(x,y)$	1/20	2/20	1/20	4/20	2/20	3/20	1/20	6/20

   (a) Find the correlation coefficient of X and Y.

   (b) Compute $\mathbf{E}[Y \mid X = k]$, $k = 0, 1, 2$, and the line $\mu_2 + \rho(\sigma_2/\sigma_1)(x - \mu_1)$. Do the points $[k, \mathbf{E}[Y \mid X = k]$, $k = 0, 1, 2$, lie on this line?

8. What do the following covariances tell you about the relationships between X and Y?

   (a) $\mathbf{COV}(X, Y) = +0.9$.

   (b) $\mathbf{COV}(X, Y) = 0$.

   (c) $\mathbf{COV}(X, Y) = -0.6$.

Section 2.5

9. Show that the random variables X_1 and X_2 with joint pmf

$$p(x_1, x_2) = \begin{cases} 1/32, & \{(x_1, x_2)\colon (0,0); (0,2); (3,0); (3,2)\} \\ 2/32, & \{(x_1, x_2)\colon (0,1); (3,1)\} \\ 3/32, & \{(x_1, x_2)\colon (1,0); (1,2); (2,0); (2,2)\} \\ 6/32, & \{(x_1, x_2)\colon (1,1); (2,1)\}. \end{cases}$$

   are independent.

10. Let X_1 and X_2 be random variables with joint pdf

$$f(x_1, x_2) = \begin{cases} \frac{1}{8} x_1 e^{-x_2}, & 0 < x_1 < 4,\ 0 < x_2 < \infty \\ 0, & \text{otherwise} \end{cases}$$

   Are X_1 and X_2 dependent or independent?

11. Let X_1 and X_2 be random variables with joint pdf

$$f(x_1, x_2) = \begin{cases} x_1 e^{-x_2}, & 0 < x_1 < x_2 < \infty \\ 0, & \text{otherwise.} \end{cases}$$

   Are X_1 and X_2 dependent or independent?

12. Explain the difference between mutually independent and pairwise independent. Which implies the other?

Section 2.6

13. Let X_1, X_2, X_3, X_4 be continuous random variables with joint pdf

$$f(x_1, x_2, x_3, x_4) = \begin{cases} \frac{4}{3} x_1 x_2^2 e^{-2x_3 - x_4}, & 0 < x_1 < 3,\ \ 0 < x_2 < 1,\ \ 0 < x_3,\ \ 0 < x_4 \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Compute $\mathsf{P}\left[X_4 < X_1 < X_2\right]$.
   (b) Find $\mathsf{P}\left[X_1 < X_2 \mid X_1 < 2X_2\right]$.
   (c) Find the marginal distribution of X_2, X_4.
   (d) Find the marginal distribution of X_1, X_2, X_4.

   Note: On an exam, you would see a maximum of 3 random variables. If you can work with 4 random variables on a study guide, working with 3 random variables should be easier.

Section 2.8

14. Let $X_1, \ldots, X_n$ be iid random variables with common mean μ and variance σ^2. Define $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$. Find $\mathbf{E}[\bar{X}]$ and $\mathbf{V}[\bar{X}]$.

15. Let X and Y be random variables with $\mu_1 = 1, \mu_2 = 4, \sigma_1^2 = 4, \sigma_2^2 = 6, \rho = \frac{1}{2}$. Find the mean and variance of the random variable $Z = 3X - 2Y$.

16. Let X_1 and X_2 be independent random variables with nonzero variances. Find the correlation coefficient of $Y = X_1 X_2$ and X_1 in terms of the means and variances of X_1 and X_2.

Section 3.1

17. Consider a standard deck of 52 cards. Let X equal the number of aces in a sample of size 2.

   (a) If the sampling is with replacement, obtain the pmf of X.
   (b) If the sampling is without replacement, obtain the pmf of X.

18. A traffic control engineer reports that 75% of the vehicles passing through a checkpoint are from within the state. What is the probability that fewer than 4 of the next 9 vehicles are from out of state? On average, how many cars will pass through the checkpoint? What is the variance?

19. Biologists doing studies in a particular environment often tag and release subjects in order to estimate the size of a population or the prevalence of certain features in the population. Ten animals of a certain population thought to be extinct (or near extinction) are caught, tagged, and released in a certain region. After a period of time, a random sample of 15 of this type of animal is selected in the region. What is the probability that 5 of those selected are tagged if there are 25 animals of this type in the region? On average, how many animals caught are tagged? What is the variance?

20. What is the probability that a waitress will refuse to serve alcoholic beverages to only 2 minors if she randomly checks the IDs of 5 among 9 students, 4 of whom are minors? On average, how many minors will the waitress refuse to serve? What is the variance?

21. It is known that 60% of mice inoculated with a serum are protected form a certain disease. If 5 mice are inoculated, find the probability that

    (a) none contracts the disease

    (b) fewer than 2 contract the disease

    (c) more than 3 contract the disease

22. The probability that a person living in a certain city owns a cat is estimated to be 0.4. Find the probability that the tenth person randomly interviewed in that city is the third one to own a cat.

23. It is known that 3% of people whose luggage is screened at an airport have questionable objects in their luggage. What is the probability that a string of 15 people pass through screening successfully before an individual is caught with a questionable object?

<u>Section 3.2</u>

24. On average, 3 traffic accidents per month occur at a certain intersection. What is the probability that at any given month at this intersection

    (a) exactly 5 accidents will occur?

    (b) fewer than 3 accidents will occur?

    (c) at least 2 accidents will occur?

25. A certain area of the eastern United States is, on average, hit by 6 hurricanes a year. Find the probability that in a given year that area will be hit by

    (a) fewer than 4 hurricanes.

    (b) anywhere from 6 to 8 hurricanes, inclusive.

26. On the average, a grocer sells three of a certain article per week. How many of these should he have in stock so that the chance of his running out within a week is less than 0.01? Assume a Poisson Distribution.

Moment Generating Functions

27. Find moment generating functions for the following probability distributions.

(a) Let X be a random variable and n a positive integer. Let $0 < p < 1$. The pmf of X is given by

$$p(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x = 0, 1, 2, \ldots, n \\ 0, & \text{otherwise.} \end{cases}$$

(b) Let X be a random variable and $\lambda > 0$ be a constant. The pmf of X is given by

$$p(x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!}, & x = 0, 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

(c) Let X be a random variable and $0 < p < 1$. The pmf of X is given by

$$p(x) = \begin{cases} (1-p)^{x-1}p, & x = 1, 2, 3, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

(d) Let X be a random variable and $a < b$ be constants. The pdf of X is given by

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise.} \end{cases}$$

(e) Let X be a random variable, $-\infty < \mu < \infty$ a constant, and $\sigma^2 > 0$ a constant. The pdf of X is given by

$$f(x) = \begin{cases} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, & -\infty < x < \infty \\ 0, & \text{otherwise.} \end{cases}$$

(f) Let X be a random variable, $\alpha > 0$ a constant, and $\theta > 0$ a constant. Let $\Gamma(\alpha)$ be a Gamma Function evaluated at α. The pdf of X is given by

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}, & 0 \le x < \infty \\ 0, & \text{otherwise.} \end{cases}$$

(g) Let X be a random variable. The pdf of X is given by

$$f(x) = \begin{cases} \frac{4}{255} x^3, & -1 < x < 4 \\ 0, & \text{otherwise.} \end{cases}$$

28. Let X_1 and X_2 be independent random variables. The pdf of X_1 is

$$f_1(x_1) = \begin{cases} \frac{1}{\Gamma(2)\left(\frac{1}{2}\right)^2} x e^{-2x}, & 0 \le x_1 < \infty \\ 0, & \text{otherwise.} \end{cases}$$

The pdf of X_2 is

$$f_2(x_2) = \begin{cases} \frac{1}{\Gamma(4)\left(\frac{1}{2}\right)^4} x^3 e^{-2x}, & 0 \le x_2 < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Find the distribution of $Y = X_1 + X_2$ using MGFs.

29. Let X_1 and X_2 be independent random variables, such that

$$p_1(x_1) = \begin{cases} \left(\frac{9}{10}\right)^{x-1} \left(\frac{1}{10}\right), & x_1 = 1, 2, 3, \ldots \\ 0, & \text{otherwise} \end{cases}$$

and

$$p_2(x_2) = \begin{cases} \left(\frac{3}{10}\right)^{x-1} \left(\frac{7}{10}\right), & x_2 = 1, 2, 3, \ldots \\ 0, & \text{otherwise} \end{cases}.$$

Use MGFs to find the pdf of $Y = X_1 + X_2$.

30. Suppose X_1 and X_2 are random variables such that their joint pdf is

$$f(x_1, x_2) = \begin{cases} x_1 e^{-x_2}, & 0 < x_1 < x_2 < \infty \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the moment generating function of X_1 and X_2, $M(t_1, t_2)$.

(b) Find the marginal distributions of X_1 and X_2.

(c) Find the moment generating function of X_1.

(d) Find the moment generating function of X_2.

Solutions

Section 2.1

1. Suppose you are given the following joint distribution for X_1 and X_2:

$$p_{1,2}(x_1,x_2) = \begin{cases} \frac{x_1+x_2}{30}, & x_1 = 0,1,2,3;\;\; x_2 = 0,1,2, \\ 0, & \text{otherwise,} \end{cases}$$

(a) Find $\mathsf{P}[X_1 \le 1, X_2 > 0]$
Solution:

$$\begin{aligned} \mathsf{P}[X_1 \le 1, X_2 > 0] &= \mathsf{P}[X_1 = 0, X_2 = 1] + \mathsf{P}[X_1 = 0, X_2 = 2] + \mathsf{P}[X_1 = 1, X_2 = 1] + \mathsf{P}[X_1 = 1, X_2 = 2] \\ &= p(0,1) + p(0,2) + p(1,1) + p(1,2) \\ &= \frac{1}{30} + \frac{2}{30} + \frac{2}{30} + \frac{3}{30} \\ &= \frac{8}{30} = \boxed{\frac{4}{15}}. \end{aligned}$$

(b) Find $\mathsf{P}[X_1 > X_2]$.
Solution:

$$\begin{aligned} \mathsf{P}[X_1 > X_2] &= \mathsf{P}[X_1 = 3, X_2 = 0] + \mathsf{P}[X_1 = 3, X_2 = 1] + \mathsf{P}[X_1 = 3, X_2 = 2] \\ &\quad + \mathsf{P}[X_1 = 2, X_2 = 0] + \mathsf{P}[X_1 = 2, X_2 = 1] \\ &\quad + \mathsf{P}[X_1 = 1, X_2 = 0] \\ &= p(3,0) + p(3,1) + p(3,2) + p(2,0) + p(2,1) + p(1,0) \\ &= 3/30 + 4/30 + 5/30 + 2/30 + 3/30 + 1/30 \\ &= \boxed{18/30 = 3/5}. \end{aligned}$$

(c) Find $F_1(x_1)$, the CDF of X_1.
Solution:

$$\begin{aligned} \mathsf{P}\left[X_1 \le x_1, -\infty < X_2 < \infty\right] &= \sum_{k=0}^{x_1} \sum_{x_2=0}^{2} \frac{k + x_2}{30} = \sum_{k=0}^{x_1} \left(\frac{k}{30} + \frac{k+1}{30} + \frac{k+2}{30} \right) \\ &= \sum_{k=0}^{x_1} \frac{1}{10}(k+1) = \frac{1}{10} \sum_{k=0}^{x_1} (k+1) = \frac{(x_1+1)(x_1+2)}{10(2)} \end{aligned}$$

$$\boxed{F_1(x_1) = \begin{cases} 0, & x_1 = \ldots, -2, -1 \\ \frac{(x_1+1)(x_1+2)}{20}, & x_1 = 0,1,2,3 \\ 1, & x_1 = 4,5,6,\ldots. \end{cases}}$$

(d) Make a table listing the marginal distribution of X. (Hint: It may help to make a table displaying the joint probability distribution of X and Y.)

Solution:

The marginal distribution of X are the column totals. Let $p(x)$ be the marginal distribution of X.

		x_1			
$p(x_1, x_2)$		0	1	2	3
	0	0	1/30	2/30	3/30
x_2	1	1/30	2/30	3/30	4/30
	2	2/30	3/30	4/30	5/30

x_1	0	1	2	3
$p_1(x_1)$	$3/30 = 1/10$	$6/30 = 1/5$	$9/30 = 3/10$	$12/30 = 2/5$

(e) Find $\mathbf{E}(X_1X_2)$.
Solution:

$$\begin{aligned}
\mathbf{E}(X_1X_2) &= \sum_{x_1}\sum_{x_2} x_1x_2\, p(x_1, x_2) \\
&= (0)(0)(0) + (0)(1)(1/30) + (0)(2)(2/30) \\
&\quad + (1)(0)(1/30) + (1)(1)(2/30) + (1)(2)(3/30) \\
&\quad + (2)(0)(2/30) + (2)(1)(3/30) + (2)(2)(4/30) \\
&\quad + (3)(0)(3/30) + (3)(1)(4/30) + (3)(2)(5/30) \\
&= 0 + 0 + 0 + 0 + 2/30 + 6/30 + 0 + 6/30 + 16/30 + 0 + 12/30 + 30/30 \\
&= \boxed{\frac{36}{15} = 2.4}.
\end{aligned}$$

(f) Find $\mathbf{E}(X_1)$.

Solution:

Find $\mathbf{E}(X_1)$ using the marginal distribution of X_1.

$$\mathbf{E}(X_1) = \sum_{x_1} x_1\, p(x_1) = (0)(1/10) + 1(1/5) + 2(3/10) + 3(2/5) = \boxed{2}.$$

2. Let X_1 and X_2 be random variables. Their joint distribution, $f(x_1, x_2)$ is given by

$$f(x,y) = \begin{cases} 10x_1x_2^2, & 0 < x_1 < x_2 < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $\mathsf{P}[X_1 < 0.25, 0.5 < X_2 < 1]$.
Solution:

$$\begin{aligned} \mathsf{P}[X_1 < 0.25, 0.5 < X_2 < 1] &= \int_{0.5}^{1}\int_{0}^{0.25} 10x_1x_2^2dx_1dx_2 = \int_{0.5}^{1} 5x_1^2x_2^2|_0^{0.25}dx_2 = \int_{0.5}^{1} 5(0.25)^2x_2^2dx_2 \\ &= \int_{0.5}^{1}\frac{5}{16}x_2^2dx_2 = \frac{5}{48}x_2^3\Big|_{0.5}^{1} = \frac{5}{48}\left[1^3 - \left(\frac{1}{2}\right)^3\right] \\ &= \frac{5}{48}\left[1-\frac{1}{8}\right] = \frac{5}{48}\left[\frac{7}{8}\right] = \boxed{\frac{35}{384}}. \end{aligned}$$

(b) Find $F_2(x_2)$, the CDF of X_2.
Solution:

$$\begin{aligned} \mathsf{P}[-\infty < X_1 < \infty, -\infty < X_2 < x_2] &= \int_0^{x_2}\int_0^k 10x_1k^2\,dx_1dk; \text{ note that } 0 < x_1 < k < 1 \\ &= \int_0^{x_2}\left(5x_1^2k^2\big|_0^k\right)dk = \int_0^{x_2} 5k^4dk = k^5\big|_0^{x_2} = x_2^5 \end{aligned}$$

$$\boxed{F_2(x_2) = \begin{cases} 0, & x_2 \le 0 \\ x_2^5 & 0 < x_2 < 1 \\ 1, & 1 \le x_2. \end{cases}}$$

(c) Find the marginal distribution of X_2.

Solution:

By differentiating the CDF:

$$\boxed{f(x_2) = \begin{cases} 5x_2^4, & 0 < x_2 < 1 \\ 0, & \text{otherwise.} \end{cases}}$$

Using the joint pdf:

$$f(x_2) = \int_0^{x_2} 10x_1x_2^2dx_1 = 5x_1^2x_2^2|_0^{x_2} = \boxed{\begin{cases} 5x_2^4, & 0 < x_2 < 1 \\ 0, & \text{otherwise.} \end{cases}}$$

(d) Find $\mathbf{E}(X_1X_2)$.
Solution:

$$\mathbf{E}(X_1X_2) = \int_{x_2}\int_{x_1} x_1x_2 f(x_1,x_2)dx_1dx_2 = \int_0^1\int_0^{x_2} x_1x_2\left(10x_1x_2^2\right)dx_1dx_2$$

$$= \int_0^1\int_0^{x_2} 10x_1^2x_2^3dx_1dx_2 = \int_0^1 \left.\frac{10x_1^3x_2^3}{3}\right|_0^{x_2} dx_2$$

$$= \int_0^1 \frac{10x_2^6}{3}dx_2 = \left.\frac{10x_2^7}{21}\right|_0^1 = \boxed{\frac{10}{21}}.$$

(e) Find the marginal distribution for X_1.

Solution:
The marginal distribution for X_1 is given by $f(x_1)$.

$$f(x_1) = \int_{x_1}^1 10x_1x_2^2dx_2 = 10x\int_{x_1}^1 x_2^2dx_2 = \left.\frac{10x_1x_2^3}{3}\right|_{x_1}^1 = \frac{10x_1(1)^3}{3} - \frac{10x_1(x_1^3)}{3}$$

$$= \boxed{\frac{10}{3}x_1\left(1-x_1^3\right),\ \ 0 < x_1 < 1}.$$

(f) Find $\mathbf{E}(X_1)$.
Solution:

$$\mathbf{E}(X_1) = \int\int g(X_1,X_2)f(x_1,x_2)\,dx_1dx_2 = \int_0^1 x_1 f(x_1)\,dx_1 = \int_0^1 x_1\left[\frac{10}{3}x_1\left(1-x_1^3\right)\right]dx_1$$

$$= \int_0^1\left(\frac{10}{3}x_1^2 - \frac{10}{3}x_1^5\right)dx_1 = \left.\frac{10x_1^3}{9} - \frac{10x^6}{18}\right|_0^1 = \frac{10}{9} - \frac{10}{18} = \boxed{\frac{10}{18} = \frac{5}{9}}.$$

(g) Find $\mathbf{E}(-5X_1)$.
Solution:

$$\mathbf{E}(-5X_1) = \int_0^1 -5x_1\left[\frac{10}{3}x_1\left(1-x_1^3\right)\right]dx_1 = -5\int_0^1 x_1\left[\frac{10}{3}x_1\left(1-x_1^3\right)\right]dx_1 = -5\ \mathbf{E}(X_1) = \boxed{\frac{-25}{9}}.$$

Section 2.2 & 2.7

3. Let X_1 and X_2 be two random variables with joint probability distribution

$$p(x_1, x_2) = \begin{cases} (1-p_1)^{x_1-1} p_1 (1-p_2)^{x_2-1} p_2, & x_1 = 1, 2, \ldots;\ x_2 = 1, 2, \ldots;\ 0 < p_1 \neq p_2 < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find the distribution of $Y_1 = X_1 + X_2$.

Solution:

Recognize that the joint distribution of X_1 and X_2 is discrete. Also, we need to define a second random variable, $Y_2 = X_2$. We solve for the x's.

$$x_2 = y_2; \qquad x_1 = y_1 - y_2.$$

Make sure to identify $\mathscr{D}_{\vec{Y}}$. At first glance, it appears that $\mathscr{D}_{X_2} = \mathscr{D}_{Y_2}$ but we need to be careful. We must also satisfy the conditions for X_1. For $Y_1 = X_1 + X_2$, we see that the smallest value Y_1 can take is 2 because $\min Y_1 = \min X_1 + X_2 = \min X_1 + \min X_2 = 1 + 1 = 2$. We also see that the smallest value Y_2 can take is 1 because $\min Y_2 = \min X_2 = 1$. We also have the restriction that Y_1 and Y_2 must be integers, and in particular, Y_2 must be a positive integer greater than or equal to 1. However, we still have not identified the maximum value that Y_2 can take.

From $x_1 = y_1 - y_2$, we have the restriction that

$$1 \leq x_1 = y_1 - y_2 \Rightarrow 1 \leq y_1 - y_2 \Rightarrow y_2 \leq y_1 - 1$$

This means in order for X_1 to be a positive number, we have to bound Y_2 above by $Y_1 - 1$. *IF* we started with the space $\mathscr{D}_{X_1} = \{0, 1, 2, \ldots\}$ and $\mathscr{D}_{X_2} = \{0, 1, 2, \ldots\}$, then we could just say that the largest value Y_2 can take is the same as Y_1, because we allow X_1 to be equal to 0. The problem in this case is that the smallest value X_1 can take is 1, which means that we can never allow Y_1 and Y_2 to take on the same value simultaneously (or else X_1 would be equal to 0, which cannot happen). However, we have no such (upper bound) restriction of what Y_1 can be. Therefore,

$$\mathscr{D}_{Y_1} = \{2, 3, \ldots\} \qquad AND \qquad \mathscr{D}_{Y_2} = \{1, 2, \ldots, y_1 - 1\}.$$

By our General Technique for Discrete Transformations, we have the joint distribution of Y_1 and Y_2 as

$$\begin{aligned} p_{Y_1,Y_2}(y_1, y_2) &= p_{X_1,X_2}(y_1 - y_2, y_2) \\ &= \begin{cases} (1-p_1)^{y_1-y_2-1} p_1 (1-p_2)^{y_2-1} p_2, & y_2 = 1, 2, \ldots, y_1 - 1;\ y_1 = 2, 3, \ldots;\ 0 < p_1 \neq p_2 < 1 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

We want to find the marginal distribution of Y_1 to answer our original question. We need the finite geometric series:

$$\sum_{k=0}^{n-1} ar^k = a \cdot \frac{1 - r^n}{1 - r}, \quad r \neq 1.$$

$$\sum_{y_2=1}^{y_1-1} (1-p_1)^{y_1-y_2-1} p_1 (1-p_2)^{y_2-1} p_2, \quad \begin{matrix} y_1 = 2, 3, \ldots \\ 0 < p_1 \neq p_2 < 1 \end{matrix} \leftarrow \text{good enough to set this up}$$

$$= p_1 p_2 (1-p_1)^{y_1} \sum_{y_2=1}^{y_1-1} (1-p_1)^{-y_2-1} (1-p_2)^{y_2-1} = p_1 p_2 (1-p_1)^{y_1} \sum_{y_2=1}^{y_1-1} (1-p_1)^{-1(y_2+1)} (1-p_2)^{y_2-1}$$

$$= p_1 p_2 (1-p_1)^{y_1} \sum_{y_2=1}^{y_1-1} \frac{(1-p_2)^{y_2-1}}{(1-p_1)^{y_2+1}}$$

$$= p_1 p_2 (1-p_1)^{y_1} \sum_{y_2=0}^{y_1-1} \frac{(1-p_2)^{y_2-1}}{(1-p_1)^{y_2+1}} - \frac{(1-p_2)^{-1}}{(1-p_1)^{1}}; \text{ add and subtract the } y_2 = 0 \text{ term}$$

$$= p_1 p_2 (1-p_1)^{y_1} \cdot \frac{(1-p_2)^{-1}}{(1-p_1)^{1}} \sum_{y_2=0}^{y_1-1} \left(\frac{1-p_2}{1-p_1}\right)^{y_2} - \frac{1}{(1-p_1)(1-p_2)}$$

$$= \frac{p_1 p_2 (1-p_1)^{y_1-1}}{1-p_2} \cdot \frac{1 - \left(\frac{1-p_2}{1-p_1}\right)^{y_1}}{1 - \left(\frac{1-p_2}{1-p_1}\right)} - \frac{1}{(1-p_1)(1-p_2)}; \text{ Geometric Series}$$

$$= \frac{p_1 p_2 (1-p_1)^{y_1-1}}{1-p_2} \cdot \frac{1 - \frac{(1-p_2)^{y_1}}{(1-p_1)^{y_1}}}{\frac{1-p_1-(1-p_2)}{1-p_1}} - \frac{1}{(1-p_1)(1-p_2)}$$

$$= \frac{p_1 p_2 (1-p_1)^{y_1-1}}{1-p_2} \cdot \frac{\frac{(1-p_1)^{y_1}-(1-p_2)^{y_1}}{(1-p_1)^{y_1}}}{\frac{p_2-p_1}{1-p_1}} - \frac{1}{(1-p_1)(1-p_2)}$$

$$= \frac{p_1 p_2 (1-p_1)^{y_1-1}}{1-p_2} \cdot \frac{(1-p_1)^{y_1}-(1-p_2)^{y_1}}{p_2-p_1} \cdot \frac{1-p_1}{(1-p_1)^{y_1}} - \frac{1}{(1-p_1)(1-p_2)}$$

$$= \frac{p_1 p_2}{1-p_2} \cdot \frac{(1-p_1)^{y_1}-(1-p_2)^{y_1}}{p_2-p_1} - \frac{1}{(1-p_1)(1-p_2)}$$

$$= \frac{p_1 p_2 \left[(1-p_1)^{y_1}-(1-p_2)^{y_1}\right](1-p_1)}{(1-p_1)(1-p_2)(p_2-p_1)} - \frac{p_2-p_1}{(1-p_1)(1-p_2)(p_2-p_1)}; \text{ common denominator}$$

$$= \frac{p_1 p_2 (1-p_1) \left[(1-p_1)^{y_1}-(1-p_2)^{y_1}\right] + p_1 - p_2}{(1-p_1)(1-p_2)(p_2-p_1)}, \quad \begin{matrix} y_1 = 2, 3, \ldots \\ 0 < p_1 \neq p_2 < 1 \end{matrix}$$

$\uparrow$ good enough if you chose to simplify (assuming I had no errors above)

4. Suppose that X is a continuous random variable such that it has the pdf

$$f(x) = \begin{cases} \frac{1}{6}, & -2 \leq x \leq 4 \\ 0, & \text{otherwise.} \end{cases}$$

Define $Y = X^2$.

(a) Find the CDF of X.
Solution:

$$\mathsf{P}(X \leq x) = \int_{-2}^{x} \frac{1}{6}\, du = \frac{1}{6} u \Big|_{-2}^{x} = \frac{1}{6}\left(x - (-2)\right) = \frac{1}{6}(x+2)$$

The CDF of X is:

$$\boxed{F_X(x) = \begin{cases} 0, & x < -2 \\ \frac{1}{6}(x+2), & -2 \leq x < 4 \\ 1, & 4 \leq x. \end{cases}}$$

(b) Find the CDF of Y.

Solution:
We know that since $x \in [-2, 4]$, then $y \in [0, 16]$. We can split these intervals into 2 portions:

i. $x \in [-2, 2]$ so $y \in [0, 4]$;
ii. $x \in [2, 4]$ so $y \in [4, 16]$.

For $y < 0$, we have $F_Y(y) = 0$.
For $y \in [0, 4]$, we have

$$\begin{aligned} F_Y(y) &= \mathsf{P}(Y \leq y) = \mathsf{P}\left(X^2 \leq y\right) = \mathsf{P}\left(-\sqrt{y} < X < \sqrt{y}\right) \\ &= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{6}\, dx = F_X\left(\sqrt{y}\right) - F_X\left(-\sqrt{y}\right) \\ &= \frac{1}{6}\left(\sqrt{y} + 2\right) - \frac{1}{6}\left(-\sqrt{y} + 2\right) \\ &= \frac{1}{6}\left[\sqrt{y} + 2 + \sqrt{y} - 2\right] = \frac{1}{6} \cdot 2\sqrt{y} = \frac{1}{3}\sqrt{y}. \end{aligned}$$

For $y \in [4, 16)$, we have

$$\begin{aligned} F_Y(y) &= \mathsf{P}(Y \leq y) = \mathsf{P}\left(X^2 \leq y\right) \\ &= \mathsf{P}\left(X \leq \sqrt{y}\right);\ \text{since } X \text{ is always positive on this interval for } y \\ &= \int_{-2}^{\sqrt{y}} \frac{1}{6}\, dx = F_X\left(\sqrt{y}\right) = \frac{1}{6}\left(\sqrt{y} + 2\right) \\ &\overset{OR}{=} \frac{x}{6}\Big|_{-2}^{\sqrt{y}} = \frac{\sqrt{y}}{6} - \frac{-2}{6} = \frac{1}{6}\left(\sqrt{y} + 2\right) \end{aligned}$$

Note that the lower bound on the integrand is -2. That is because we are looking at cumulative information from the point where X starts ($-2 \leq x \leq 4$), up until $\sqrt{y}$.

For $y \geq 16$, we have $F_Y(y) = 1$.
The CDF of Y is:

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{1}{3}\sqrt{y}, & 0 \leq y < 4 \\ \frac{1}{6}\left(\sqrt{y} + 2\right), & 4 \leq y < 16 \\ 1, & y \geq 16. \end{cases}$$

(c) Find the PDF of Y.

Solution:
Find the first derivative of the CDF of Y with respect to y. The PDF of Y is:

$$f_Y(y) = \begin{cases} \frac{1}{6\sqrt{y}}, & 0 \leq y < 4 \\ \frac{1}{12\sqrt{y}}, & 4 \leq y < 16 \\ 0, & \text{otherwise.} \end{cases}$$

5. Let X_1 and X_2 be two continuous random variables with joint probability distribution

$$f(x_1, x_2) = \begin{cases} \frac{1}{\pi}, & 0 < x_1^2 + x_2^2 < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Define $Y_1 = X_1^2 + X_2^2$ and $Y_2 = \frac{X_1^2}{X_1^2 + X_2^2}$. Find the joint pdf of Y_1 and Y_2.

Solution:

- Solve for x_1 and x_2.

$$x_1^2 = y_1 y_2$$
$$x_2^2 = y_1 - x_1^2 = y_1 - y_1 y_2 = y_1 (1 - y_2).$$

- Find the space for Y, $\mathscr{D}_Y$ to know the values of x_1 and x_2 exactly.
  Note that $0 < x_1^2 + x_2^2 < 1$ corresponds to values inside of the unit circle. I label the different quadrants in the usual way.
  - In Quadrant I: $x_1 = \sqrt{y_1 y_2}$ and $x_2 = \sqrt{y_1(1 - y_2)}$.
  - In Quadrant II: $x_1 = -\sqrt{y_1 y_2}$ and $x_2 = \sqrt{y_1(1 - y_2)}$.
  - In Quadrant III: $x_1 = -\sqrt{y_1 y_2}$ and $x_2 = -\sqrt{y_1(1 - y_2)}$.
  - In Quadrant IV: $x_1 = \sqrt{y_1 y_2}$ and $x_2 = -\sqrt{y_1(1 - y_2)}$.

  We then have

$$0 < x_1^2 + x_2^2 < 1 \Rightarrow 0 < y_1 y_2 + y_1 - y_1 y_2 < 1 \Rightarrow 0 < y_1 < 1.$$

We also have, from the unit circle, that

$$\begin{aligned}
-1 < x_2 < 1 &\Rightarrow 0 < x_2^2 < 1 \\
&\Rightarrow 0 < y_1(1-y_2) < 1 \\
&\Rightarrow 0 < 1-y_2 < 1; \text{ since we already know } 0 < y_1 < 1 \\
&\Rightarrow 0 < y_2 < 1.
\end{aligned}$$

Therefore,

$$\boxed{0 < y_1 < 1; \qquad 0 < y_2 < 1}.$$

- Find J and $|J|$.
  - In Quadrant I:

$$\begin{aligned}
J &= \begin{vmatrix} \frac{1}{2}\sqrt{\frac{y_2}{y_1}} & \frac{1}{2}\sqrt{\frac{y_1}{y_2}} \\ \frac{1}{2}\sqrt{\frac{1-y_2}{y_1}} & -\frac{1}{2}\sqrt{\frac{y_1}{1-y_2}} \end{vmatrix} = -\frac{1}{4}\sqrt{\frac{y_2}{1-y_2}} - \frac{1}{4}\sqrt{\frac{1-y_2}{y_2}} \\
&= \frac{-y_2-(1-y_2)}{4\sqrt{y_2\,(1-y_2)}} = \frac{-1}{4\sqrt{y_2\,(1-y_2)}} \\
|J| &= \frac{1}{4\sqrt{y_2\,(1-y_2)}}.
\end{aligned}$$

  - In Quadrant IV:

$$J = \begin{vmatrix} \frac{1}{2}\sqrt{\frac{y_2}{y_1}} & \frac{1}{2}\sqrt{\frac{y_1}{y_2}} \\ -\frac{1}{2}\sqrt{\frac{1-y_2}{y_1}} & \frac{1}{2}\sqrt{\frac{y_1}{1-y_2}} \end{vmatrix} = \frac{1}{4}\sqrt{\frac{y_2}{1-y_2}} + \frac{1}{4}\sqrt{\frac{1-y_2}{y_2}} = \frac{1}{4\sqrt{y_2\,(1-y_2)}} = |J|.$$

  - In Quadrant II:

$$|J| = \frac{1}{4\sqrt{y_2\,(1-y_2)}}.$$

  - In Quadrant III:

$$|J| = \frac{1}{4\sqrt{y_2\,(1-y_2)}}.$$

- Find $f(y_1, y_2)$. Note that $f(y_1, y_2)$ is the sum of the four quadrants, which all have the same pdf for (X_1, X_2) and the same Jacobian.

$$\boxed{f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 4 \cdot \frac{1}{\pi} \cdot |J| = \frac{1}{\pi\sqrt{y_2(1-y_2)}}, & \begin{matrix} 0 < y_1 < 1 \\ 0 < y_2 < 1 \end{matrix} \\ 0, & \text{otherwise.} \end{cases}}$$

Section 2.3

6. Let X_1 and X_2 be continuous random variables with joint probability distribution

$$f(x_1, x_2) = \begin{cases} \frac{5}{16} x_1 x_2^2, & 0 < x_1 < x_2 < 2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $\mathbf{E}[X_1 X_2]$.
Solution:

$$\begin{aligned} \mathbf{E}[X_1 X_2] &= \int_0^2 \int_0^{x_2} x_1 x_2 \cdot \frac{5}{16} x_1 x_2^2 \, dx_1 \, dx_2 = \int_0^2 \int_0^{x_2} \frac{5}{16} x_1^2 x_2^3 \, dx_1 \, dx_2 \\ &= \int_0^2 \frac{5}{16} \cdot \frac{x_1^3}{3} \cdot x_2^3 \bigg|_0^{x_2} dx_2 = \int_0^2 \frac{5}{48} x_2^6 \, dx_2 = \frac{5}{48} \left[\frac{x_2^7}{7} \bigg|_0^2 \right] \\ &= \frac{5}{48} \cdot \frac{128}{7} = \boxed{\frac{40}{21}}. \end{aligned}$$

(b) Find the marginal distribution of X_2.
Solution:

$$\int_0^{x_2} \frac{5}{16} x_1 x_2^2 \, dx_1 = \frac{5}{16} x_2^2 \int_0^{x_2} x_1 \, dx_1 = \frac{5}{16} x_2^2 \cdot \frac{x_1^2}{2} \bigg|_0^{x_2} = \frac{5}{16} x_2^2 \left(\frac{x_2^2}{2} - 0 \right) = \frac{5}{32} x_2^4.$$

The marginal distribution of X_2 is:

$$\boxed{f_2(x_2) = \begin{cases} \frac{5}{32} x_2^4, & 0 < x_2 < 2 \\ 0, & \text{otherwise.} \end{cases}}$$

(c) Find the conditional distribution of X_1, given $X_2 = x_2$.
Solution:

$$f_{1|2}(X_1 \mid X_2 = x_2) = \frac{f(x_1, x_2)}{f(x_2)} = \frac{\frac{5}{16} x_1 x_2^2}{\frac{5}{32} x_2^4} = \frac{2x_1}{x_2^2}.$$

The conditional distribution of X_1, given $X_2 = x_2$ is

$$\boxed{f_{1|2}(x_1 \mid X_2 = x_2) = \begin{cases} \frac{2x_1}{x_2^2}, & 0 < x_1 < 2 \\ 0, & \text{otherwise.} \end{cases}}$$

(d) Find $\mathsf{P}\left[0 < X_1 < 1 \mid X_2 = \frac{3}{2}\right]$.
Solution:

$$\mathsf{P}\left[0 < X_1 < 1 \mid X_2 = \frac{3}{2}\right] = \int_0^1 \frac{2x_1}{\left(\frac{3}{2}\right)^2} \, dx_1 = \frac{8}{9} \cdot \frac{x_1^2}{2} \bigg|_0^1 = \frac{8}{9} \cdot \frac{1}{2} = \boxed{\frac{4}{9}}.$$

(e) Find $\mathsf{P}[0 < X_1 < 1]$.

Solution:

Option 1: Solve using the joint pdf.

$$\begin{aligned}\mathsf{P}[0 < X_1 < 1] &= \int_0^1 \int_{x_1}^2 \frac{5}{16} x_1 x_2^2 \, dx_2 \, dx_1 = \int_0^1 \frac{5}{16} x_1 \cdot \left.\frac{x_2^3}{3}\right|_{x_1}^2 \, dx_1 \\ &= \int_0^1 \frac{5}{16} x_1 \left(\frac{8}{3} - \frac{x_1^3}{3}\right) dx_1 = \int_0^1 \left(\frac{5}{6} x_1 - \frac{5}{48} x_1^4\right) dx_1 \\ &= \left.\frac{5}{6} \cdot \frac{x_1^2}{2} - \frac{5}{48} \cdot \frac{x_1^5}{5}\right|_0^1 = \frac{5}{6} \cdot \frac{1}{2} - \frac{5}{48} \cdot \frac{1}{5} = \boxed{\frac{19}{48}}.\end{aligned}$$

Option 2: First find the marginal distribution of X_1, then find the probability.

$$\begin{aligned}\int_{x_1}^2 \frac{5}{16} x_1 x_2^2 \, dx_2 &= \frac{5}{16} x_1 \int_{x_1}^2 x_2^2 \, dx_2 = \frac{5}{16} x_1 \cdot \left.\frac{x_2^3}{3}\right|_{x_1}^2 = \frac{5}{16} x_1 \left(\frac{8}{3} - \frac{x_1^3}{3}\right) \\ &= \frac{5}{48} x_1 \left(8 - x_1^3\right) = \frac{5}{6} x_1 - \frac{5}{48} x_1^4.\end{aligned}$$

The marginal distribution of X_1 is:

$$f_1(x_1) = \begin{cases} \frac{5}{48} x_1 \left(8 - x_1^3\right), & 0 < x_1 < 2 \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$\mathsf{P}[0 < X_1 < 1] = \int_0^1 \left(\frac{5}{6} x_1 - \frac{5}{48} x_1^4\right) dx_1 = \left.\frac{5}{6} \cdot \frac{x_1^2}{2} - \frac{5}{48} \cdot \frac{x_1^5}{5}\right|_0^1 = \frac{5}{6} \cdot \frac{1}{2} - \frac{5}{48} \cdot \frac{1}{5} = \boxed{\frac{19}{48}}.$$

(f) Find $\mathbf{E}[X_1]$.

Solution:

Option 1: Solve using the joint pdf.

$$\begin{aligned}\mathbf{E}[X_1] &= \int_0^2 \int_0^{x_2} x_1 \cdot \frac{5}{16} x_1 x_2^2 \, dx_1 \, dx_2 = \int_0^2 \int_0^{x_2} \frac{5}{16} x_1^2 x_2^2 \, dx_1 \, dx_2 \\ &= \int_0^2 \left.\frac{5}{16} \cdot \frac{x_1^3}{3} \cdot x_2^2\right|_0^{x_2} dx_2 = \int_0^2 \frac{5}{48} x_2^5 \, dx_2 = \left.\frac{5}{48} \cdot \frac{x_2^6}{6}\right|_0^2 = \frac{5}{48} \cdot \frac{32}{3} = \boxed{\frac{10}{9}}.\end{aligned}$$

Option 2: If you found the marginal distribution of X_1 in part (d), then you can just find the expected value directly from it.

$$\begin{aligned}\mathbf{E}[X_1] &= \int_0^2 x_1 \left(\frac{5}{6} x_1 - \frac{5}{48} x_1^4\right) dx_1 = \int_0^2 \left(\frac{5}{6} x_1^2 - \frac{5}{48} x_1^5\right) dx_2 \\ &= \left.\frac{5}{6} \cdot \frac{x_1^3}{3} - \frac{5}{48} \cdot \frac{x_1^6}{6}\right|_0^2 = \frac{5}{6} \cdot \frac{8}{3} - \frac{5}{48} \cdot \frac{32}{3} = \boxed{\frac{10}{9}}.\end{aligned}$$

(g) Find $\mathbf{V}[X_1]$.

Solution:

Option 1: Solve using the joint pdf.

$$\begin{aligned}
\mathbf{E}\left[X_1^2\right] &= \int_0^2 \int_0^{x_2} x_1^2 \cdot \frac{5}{16} x_1 x_2^2 \, dx_1 \, dx_2 = \int_0^2 \int_0^{x_2} \frac{5}{16} x_1^3 x_2^2 \, dx_2 \, dx_2 \\
&= \int_0^2 \frac{5}{16} \cdot \frac{x_1^4}{4} x_2^2 \Big|_0^{x_2} \, dx_2 = \int_0^2 \frac{5}{64} x_2^6 \, dx_2 \\
&= \frac{5}{64} \cdot \frac{x_2^7}{7} \Big|_0^2 = \frac{5}{64} \cdot \frac{128}{7} = \frac{10}{7}, \\
\mathbf{V}[X_1] = \mathbf{E}\left[X_1^2\right] - (\mathbf{E}[X_1])^2 &= \frac{10}{7} - \left(\frac{10}{9}\right)^2 = \boxed{\frac{110}{567}}.
\end{aligned}$$

Option 2: If you found the marginal distribution of X_1 in part (d), then you can just find the variance directly from it.

$$\begin{aligned}
\mathbf{E}\left[X_1^2\right] &= \int_0^2 x_1^2 \left(\frac{5}{6} x_1 - \frac{5}{48} x_1^4\right) dx_1 = \int_0^2 \left(\frac{5}{6} x_1^3 - \frac{5}{48} x_1^6\right) dx_1 \\
&= \frac{5}{6} \cdot \frac{x_1^4}{4} - \frac{5}{48} \cdot \frac{x_1^7}{7} \Big|_0^2 = \frac{5}{6} \cdot 4 - \frac{5}{48} \cdot \frac{128}{7} = \frac{10}{7}, \\
\mathbf{V}[X_1] = \mathbf{E}\left[X_1^2\right] - (\mathbf{E}[X_1])^2 &= \frac{10}{7} - \left(\frac{10}{9}\right)^2 = \boxed{\frac{110}{567}}.
\end{aligned}$$

(h) Find the distribution of $Y = \mathbf{E}[X_1 \mid X_2]$.

Solution:

First, find the value of $\mathbf{E}[X_1 \mid X_2]$.

$$\begin{aligned}
\mathbf{E}[X_1 \mid X_2] &= \int_{-\infty}^{\infty} x_1 f_{1|2}(x_1 \mid x_2) \, dx_1 = \int_0^{x_2} x_1 \cdot \frac{2x_1}{x_2^2} \, dx_1 \\
&= \int_0^{x_2} \frac{2x_1^2}{x_2^2} \, dx_1 = \frac{2}{x_2^2} \cdot \frac{x_1^3}{3} \Big|_0^{x_2} = \frac{2}{x_2^2} \cdot \frac{x_2^3}{3} \\
&= \frac{2x_2}{3}, \quad 0 < x_2 < 2.
\end{aligned}$$

Next, find the CDF of Y.

$$\mathsf{P}[Y \le y] = \mathsf{P}[\mathbf{E}(X_1 \mid X_2) \le y] = \mathsf{P}\left[\frac{2X_2}{3} \le y\right] = \mathsf{P}\left[X_2 \le \frac{3}{2} y\right]$$

$$= \int_0^{3y/2} \frac{5}{32} x_2^4 \, dx_2 = \frac{5}{32} \cdot \frac{x_2^5}{5} \Big|_0^{3y/2} = \frac{5}{32} \cdot \frac{\left(\frac{3y}{2}\right)^5}{5} = \frac{1}{32} \cdot \frac{243}{32} y^5 = \frac{243}{1024} y^5.$$

Note the values Y can take.

$$0 < x_2 < 2 \Rightarrow 0 < \frac{3y}{2} < 2 \Rightarrow 0 < y < \frac{4}{3}.$$

The CDF of Y is:

$$\boxed{F_Y(y) = \begin{cases} 0, & y < 0 \\ \frac{243}{1024}y^5, & 0 \le y < \frac{4}{3} \\ 1, & \frac{4}{3} \le y. \end{cases}}$$

The PDF of Y is:

$$\boxed{f_Y(y) = \begin{cases} \frac{1215}{1024}y^4, & 0 < y < \frac{4}{3} \\ 0, & \text{otherwise.} \end{cases}}$$

(i) Find $\mathbf{E}[Y]$.

Solution:

By a theorem in the text,

$$\mathbf{E}[Y] = \mathbf{E}[\mathbf{E}(X_1 \mid X_2)] = \mathbf{E}[X_1] = \boxed{\frac{10}{9}}.$$

(j) Find $\mathbf{V}[Y]$. How does this value compare to $\mathbf{V}[X_1]$?

Solution:

$$\mathbf{E}\left[Y^2\right] = \int_0^{4/3} y^2 \cdot \frac{1215}{1024}y^4\,dy = \int_0^{4/3} \frac{1215}{1024}y^6\,dy = \frac{1215}{1024} \cdot \frac{y^7}{7}\bigg|_0^{4/3} = \frac{1215}{1024} \cdot \frac{\left(\frac{4}{3}\right)^7}{7}$$

$$= \frac{1215}{1024} \cdot \frac{16384}{7(2187)} = \frac{1215(16)}{7(2187)} = \frac{19440}{15309} = \frac{80}{63},$$

$$\mathbf{V}[Y] = \mathbf{E}\left[Y^2\right] - (\mathbf{E}[Y])^2 = \frac{80}{63} - \left(\frac{10}{9}\right)^2 = \boxed{\frac{20}{567}}.$$

Note that $\mathbf{V}[Y] = \frac{20}{567} \le \frac{110}{567} = \mathbf{V}[X_1]$, and this is also true by a theorem in the text.

Section 2.4

7. Let X and Y have the joint pmf described as follows:

(x, y)	$(0,1)$	$(0,3)$	$(0,5)$	$(1,1)$	$(1,3)$	$(1,5)$	$(2,1)$	$(2,5)$
$p(x, y)$	1/20	2/20	1/20	4/20	2/20	3/20	1/20	6/20

(a) Find the correlation coefficient of X and Y.

Solution:
First, I make a table.

		y			
$p(x,y)$		1	3	5	$p(x)$
	0	1/20	2/20	1/20	4/20
x	1	4/20	2/20	3/20	9/20
	2	1/20	0	6/20	7/20
	$p(y)$	6/20	4/20	10/20	1

$$\mu_1 = \mathbf{E}[X] = \sum_x xp(x) = 0\left(\frac{4}{20}\right) + 1\left(\frac{9}{20}\right) + 2\left(\frac{7}{20}\right) = \frac{23}{20}$$

$$\mathbf{E}\left[X^2\right] = \sum_x x^2p(x) = 0^2\left(\frac{4}{20}\right) + 1^2\left(\frac{9}{20}\right) + 2^2\left(\frac{7}{20}\right) = \frac{37}{20}$$

$$\sigma_1^2 = \mathbf{V}[X] = \mathbf{E}\left[X^2\right] - (\mathbf{E}[X])^2 = \frac{37}{20} - \left(\frac{23}{20}\right)^2 = \frac{211}{400}$$

$$\mu_2 = \mathbf{E}[Y] = \sum_y yp(y) = 1\left(\frac{6}{20}\right) + 3\left(\frac{4}{20}\right) + 5\left(\frac{10}{20}\right) = \frac{17}{5}$$

$$\mathbf{E}\left[Y^2\right] = \sum_y y^2p(y) = 1^2\left(\frac{6}{20}\right) + 3^2\left(\frac{4}{20}\right) + 5^2\left(\frac{10}{20}\right) = \frac{73}{5}$$

$$\sigma_2^2 = \mathbf{V}[Y] = \mathbf{E}\left[Y^2\right] - (\mathbf{E}[Y])^2 = \frac{73}{5} - \left(\frac{17}{5}\right)^2 = \frac{76}{25}$$

$$\begin{aligned}\mathbf{E}[XY] &= \sum_x\sum_y xyp(x,y) = \frac{87}{20} \\ &= (0)(1)\left(\frac{1}{20}\right) + (0)(3)\left(\frac{2}{20}\right) + (0)(5)\left(\frac{1}{20}\right) \\ &\quad + (1)(1)\left(\frac{4}{20}\right) + (1)(3)\left(\frac{2}{20}\right) + (1)(5)\left(\frac{3}{20}\right) \\ &\quad + (2)(1)\left(\frac{1}{20}\right) + (2)(3)(0) + (2)(5)\left(\frac{6}{20}\right)\end{aligned}$$

$$\mathbf{COV}(X,Y) = \mathbf{E}[XY] - \mu_1\mu_2 = \frac{87}{20} - \left(\frac{23}{20}\right)\left(\frac{17}{5}\right) = \frac{11}{25} = 0.44$$

$$\rho = \frac{\mathbf{COV}(X,Y)}{\sigma_1\sigma_2} = \frac{11/25}{\sqrt{211/400}\sqrt{76/25}} = \boxed{0.3475}.$$

There is a moderate positive linear relationship between X and Y.

(b) Compute $\mathbf{E}[Y \mid X = k]$, $k = 0, 1, 2$, and the line $\mu_2 + \rho(\sigma_2/\sigma_1)(x - \mu_1)$. Do the points $[k, \mathbf{E}[Y \mid X = k]$, $k = 0, 1, 2$, lie on this line?
Solution:

$$\mathbf{E}[Y \mid X = 0] = \frac{\sum_y yp(0,y)}{p(X=0)} = \frac{1(1/20) + 3(2/20) + 5(1/20)}{4/20} = \frac{3/5}{4/20} = \boxed{3},$$

$$\mathbf{E}[Y \mid X = 1] = \frac{\sum_y yp(1,y)}{p(X=1)} = \frac{1(4/20) + 3(2/20) + 5(3/20)}{9/20} = \frac{5/4}{9/20} = \boxed{\frac{25}{9}},$$

$$\mathbf{E}[Y \mid X = 2] = \frac{\sum_y yp(2,y)}{p(X=2)} = \frac{1(1/20) + 3(0) + 5(6/20)}{7/20} = \frac{31/20}{7/20} = \boxed{\frac{31}{7}}.$$

Find the line $\mu + 2 + \rho(\sigma_2/\sigma_1)(x - \mu_1)$.

$$\begin{aligned}
\mu_2 + \rho(\sigma_2/\sigma_1)(x - \mu_1) &= \frac{17}{5} + \frac{11/25}{\sqrt{211/400}\sqrt{76/25}}\left(\frac{\sqrt{76/25}}{\sqrt{211/400}}\right)\left(x - \frac{23}{20}\right) \\
&= \frac{17}{5} + \frac{22}{\sqrt{4009}}\left(4\sqrt{\frac{76}{211}}\right)\left(x - \frac{23}{20}\right) \\
&= \frac{17}{5} + \frac{176}{211}\left(x - \frac{23}{20}\right)
\end{aligned}$$

If $x = 0$, then the value of the line is

$$\frac{17}{5} + \frac{176}{211}\left(0 - \frac{23}{20}\right) = \frac{515}{211}.$$

If $x = 1$, then the value of the line is

$$\frac{17}{5} + \frac{176}{211}\left(1 - \frac{23}{20}\right) = \frac{691}{211}.$$

If $x = 2$, then the value of the line is

$$\frac{17}{5} + \frac{176}{211}\left(2 - \frac{23}{20}\right) = \frac{867}{211}.$$

These points do not lie on the line. By Theorem 2.4.1, if $\mathbf{E}[Y \mid X]$ is linear in X, then

$$\mathbf{E}[Y \mid X] = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(X - \mu_1).$$

Because this is not the case, then $\mathbf{E}[Y \mid X]$ must not be linear in X.

8. What do the following covariances tell you about the relationships between X and Y?

   (a) $\mathbf{COV}(X, Y) = +0.9$.

   Solution:

   There is a positive linear relationship between X and Y. It cannot tell us the strength of the linear relationship; we need ρ to tell us that.

   (b) $\mathbf{COV}(X, Y) = 0$.

   Solution:

   There is no linear relationship between X and Y. Note that $\mathbf{COV}(X, Y) = 0 \not\Rightarrow$ Independent, but Independent $\Rightarrow \mathbf{COV}(X, Y) = 0$.

   (c) $\mathbf{COV}(X, Y) = -0.6$.

   Solution:

   There is a negative linear relationship between X and Y. It cannot tell us the strength of the linear relationship; we need ρ to tell us that.

Section 2.5

9. Show that the random variables X_1 and X_2 with joint pmf

$$p(x_1, x_2) = \begin{cases} 1/32, & \{(x_1, x_2)\colon (0,0); (0,2); (3,0); (3,2)\} \\ 2/32, & \{(x_1, x_2)\colon (0,1); (3,1)\} \\ 3/32, & \{(x_1, x_2)\colon (1,0); (1,2); (2,0); (2,2)\} \\ 6/32, & \{(x_1, x_2)\colon (1,1); (2,1)\}. \end{cases}$$

are independent.

Solution:

First, I put the values for the pmf of X_1, X_2 into a table:

$p(x_1, x_2)$		$x_2 = 0$	$x_2 = 1$	$x_2 = 2$	$p(x_1)$
x_1	0	1/32	2/32	1/32	4/32
	1	3/32	6/32	3/32	12/32
	2	3/32	6/32	3/32	12/32
	3	1/32	2/32	1/32	4/32
$p(x_2)$		8/32	16/32	8/32	1

You could then find the marginal distributions of X_1 and X_2 and show that $p(x_1, x_2) = p(x_1)p(x_2)$.

x_1	0	1	2	3
$p(x_1)$	1/8	3/8	3/8	1/8

x_2	0	1	2
$p(x_2)$	1/4	2/4	1/4

It is clear that $p(x_1, x_2) = p(x_1)p(x_2)$.
Examples:

- $p_{1,2}(0, 0) = 1/32 = (1/8)(1/4) = p_1(0)p_2(0)$.
- $p_{1,2}(3, 1) = 2/32 = (1/8)(2/4) = p_1(3)p_2(1)$.

10. Let X_1 and X_2 be random variables with joint pdf

$$f(x_1, x_2) = \begin{cases} \frac{1}{8}x_1 e^{-x_2}, & 0 < x_1 < 4,\ 0 < x_2 < \infty \\ 0, & \text{otherwise} \end{cases}$$

Are X_1 and X_2 dependent or independent?

Solution:

Option 1:
Since $f(x_1, x_2)$ can be decomposed into a product of non-negative functions, and there the domains for X_1 and X_2 do not depend on each other, then X_1 and X_2 are independent.

Option 2:
Find the marginal distributions of X_1 and X_2 and show if $f(x_1, x_2) = f(x_1)f(x_2)$.

$$f(x_1) = \int_0^\infty \frac{1}{8}x_1 e^{-x_2}\,dx_2 = -\frac{1}{8}x_1 e^{-x_2}\Big|_0^\infty = \boxed{\frac{1}{8}x_1,\ 0 < x_1 < 4},$$
$$f(x_2) = \int_0^4 \frac{1}{8}x_1 e^{-x_2}\,dx_1 = \frac{1}{8}\cdot\frac{x_1^2}{2}e^{-x_2}\Big|_0^4 = \boxed{e^{-x_2},\ 0 < x_2 < \infty}.$$

Because

$$f(x_1)f(x_2) = \frac{1}{8}x_1 \cdot e^{-x_2} = f(x_1, x_2),$$

X_1 and X_2 are independent.

11. Let X_1 and X_2 be random variables with joint pdf

$$f(x_1, x_2) = \begin{cases} x_1 e^{-x_2}, & 0 < x_1 < x_2 < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Are X_1 and X_2 dependent or independent?

Solution:

Find the marginal distributions of X_1 and X_2 and show if $f(x_1, x_2) = f(x_1)f(x_2)$.

$$f(x_1) = \int_{x_1}^\infty x_1 e^{-x_2}\,dx_2 = -x_1 e^{-x_2}\Big|_{x_1}^\infty = \boxed{x_1 e^{-x_1},\ 0 < x_1 < \infty},$$

$$f(x_2) = \int_0^{x_2} x_1 e^{-x_2}\,dx_1 = \frac{x_1^2}{2}e^{-x_2}\Big|_0^{x_2} = \boxed{\frac{x_2^2 e^{-x_2}}{2},\ 0 < x_2 < \infty}.$$

Because

$$f(x_1)f(x_2) = x_1 e^{-x_1}\cdot\frac{x_2^2 e^{-x_2}}{2} \neq x_1 e^{-x_2} = f(x_1, x_2),$$

X_1 and X_2 are dependent.

12. Explain the difference between mutually independent and pairwise independent. Which implies the other?

Solution:

Mutually independent means that you can take any combination of random variables under consideration, and they will all be independent of each other. Pairwise independent means when you take any 2 random variables under consideration, they will be independent.
Mutually independent implies pairwise independent. Pairwise independent does not always imply mutual independence (see counterexample from in-class notes).
If X_1, X_2, X_3 are mutually independent, then so are X_1, X_2; X_1, X_3; and X_2, X_3 (all of the different possible pairs).

Section 2.6

13. Let X_1, X_2, X_3, X_4 be continuous random variables with joint pdf

$$f(x_1, x_2, x_3, x_4) = \begin{cases} \frac{4}{3} x_1 x_2^2 e^{-2x_3 - x_4}, & 0 < x_1 < 3,\ 0 < x_2 < 1,\ 0 < x_3,\ 0 < x_4 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Compute $\mathsf{P}\left[X_4 < X_1 < X_2\right]$.
Solution:

$$\begin{aligned}
\mathsf{P}\left[X_4 < X_1 < X_2\right] &= \int_0^\infty \int_0^1 \int_0^{x_2} \int_0^{x_1} \frac{4}{3} x_1 x_2^2 e^{-2x_3 - x_4} \, dx_4 \, dx_1 \, dx_2 \, dx_3 \\
&= \frac{4}{3} \left(\int_0^\infty e^{-2x_3} \, dx_3 \right) \left(\int_0^1 \int_0^{x_2} \int_0^{x_1} x_1 x_2^2 e^{-x_4} \, dx_4 \, dx_1 \, dx_2 \right) \\
&= \frac{4}{3} \left(\frac{-1}{2} e^{-2x_3} \Big|_0^\infty \right) \left(\int_0^1 \int_0^{x_2} x_1 x_2^2 \left[-e^{-x_4} \Big|_0^{x_1} \right] \, dx_1 \, dx_2 \right) \\
&= \frac{2}{3} \int_0^1 \int_0^{x_2} x_1 x_2^2 \left[1 - e^{-x_1} \right] \, dx_1 \, dx_2 \\
&= \frac{2}{3} \int_0^1 \int_0^{x_2} \left(x_1 x_2^2 - x_1 x_2^2 e^{-x_1} \right) \, dx_1 \, dx_2 \\
&= \frac{2}{3} \int_0^1 \left(\frac{x_1^2}{2} \cdot x_2^2 - x_2^2 \left[-x_1 e^{-x_1} - e^{-x_1} \right] \Big|_0^{x_2} \right) \, dx_2 \\
&= \frac{2}{3} \int_0^1 \left(\frac{1}{2} x_2^4 - x_2^2 \left[-x_2 e^{-x_2} - e^{-x_2} + e^0 \right] \right) \, dx_2 \\
&= \frac{2}{3} \int_0^1 \left(\frac{1}{2} x_2^4 + x_2^3 e^{-x_2} + x_2^2 e^{-x_2} - x_2^2 \right) \, dx_2
\end{aligned}$$

$$= \frac{2}{3}\left(\frac{1}{2}\cdot\frac{x_2^5}{5} + \left[-x_2^3e^{-x_2} - 3x_2^2e^{-x_2} - 6x_2e^{-x_2} - 6e^{-x_2}\right]\right.$$

$$\left. + \left[-x_2^2e^{-x_2} - 2x_2e^{-x_2} - 2e^{-x_2}\right] - \frac{x_2^3}{3}\Big|_0^1\right)$$

$$= \frac{2}{3}\left(\frac{1}{10} + \left[-1e^{-1} - 3(1)^2e^{-1} - 6(1)e^{-1} - 6e^{-1} - \left\{0 - 0 - 0 - 6e^0\right\}\right]\right.$$

$$\left. + \left[-(1)^2e^{-1} - 2(1)e^{-1} - 2e^{-1} - \left\{0 - 0 - 2e^0\right\}\right] - \frac{1}{3}\right)$$

$$= \frac{2}{3}\left(\frac{1}{10} - 16e^{-1} + 6 - 5e^{-1} + 2 - \frac{1}{3}\right)$$

$$= \frac{2}{3}\left(\frac{233}{30} - 21e^{-1}\right)$$

$$= \boxed{\frac{233}{45} - 14e^{-1} \approx 0.0275}.$$

(b) Find $\mathsf{P}[X_1 < X_2 \mid X_1 < 2X_2]$.
Solution:

$$\mathsf{P}[X_1 < X_2 \mid X_1 < 2X_2] = \frac{\mathsf{P}[X_1 < X_2 \cap X_1 < 2X_2]}{\mathsf{P}[X_1 < 2X_2]} = \frac{\mathsf{P}[X_1 < X_2]}{\mathsf{P}[X_1 < 2X_2]}.$$

Now, we just need to find the individual probabilities.

$$\mathsf{P}[X_1 < X_2] = \int_0^\infty \int_0^\infty \int_0^1 \int_0^{x_2} \frac{4}{3}x_1x_2^2e^{-2x_3-x_4}\,dx_1\,dx_2\,dx_3\,dx_4$$

$$= \left(\int_0^\infty e^{-x_4}\,dx_4\right)\left(\int_0^\infty e^{-2x_3}\,dx_3\right)\left(\int_0^1\int_0^{x_2}\frac{4}{3}x_1x_2^2\,dx_1\,dx_2\right)$$

$$= \left(-e^{-x_4}\big|_0^\infty\right)\left(-\frac{1}{2}e^{-2x_3}\Big|_0^\infty\right)\left(\int_0^1 \frac{4}{3}\cdot\frac{x_1^2}{2}\cdot x_2^2\Big|_0^{x_2}\,dx_2\right)$$

$$= (1)\left(\frac{1}{2}\right)\left(\frac{4}{3}\int_0^1\frac{1}{2}x_2^4\,dx_2\right) = \frac{1}{2}\left(\frac{2}{3}\cdot\frac{x_2^5}{5}\Big|_0^1\right) = \frac{1}{2}\left(\frac{2}{3}\cdot\frac{1}{5}\right) = \frac{1}{15},$$

$$
\begin{aligned}
\mathsf{P}\left[X_1 < 2X_2\right] &= \int_0^\infty \int_0^\infty \int_0^1 \int_0^{2x_2} \frac{4}{3} x_1 x_2^2 e^{-2x_3 - x_4}\, dx_1\, dx_2\, dx_3\, dx_4 \\
&= \left(\int_0^\infty e^{-x_4}\, dx_4\right)\left(\int_0^\infty e^{-2x_3}\, dx_3\right)\left(\int_0^1 \int_0^{2x_2} \frac{4}{3} x_1 x_2^2\, dx_1\, dx_2\right) \\
&= \left(-e^{-x_4}\Big|_0^\infty\right)\left(-\frac{1}{2} e^{-2x_3}\Big|_0^\infty\right)\left(\int_0^1 \frac{4}{3} \cdot \frac{x_1^2}{2} \cdot x_2^2 \Big|_0^{2x_2} dx_2\right) \\
&= (1)\left(\frac{1}{2}\right)\left(\frac{2}{3}\int_0^1 4x_2^4\, dx_2\right) = \frac{1}{2}\left(\frac{8}{3} \cdot \frac{x_2^5}{5}\Big|_0^1\right) = \frac{1}{2}\left(\frac{8}{3} \cdot \frac{1}{5}\right) = \frac{4}{15}.
\end{aligned}
$$

Therefore,

$$
\mathsf{P}\left[X_1 < X_2 \mid X_1 < 2X_2\right] = \frac{\mathsf{P}\left[X_1 < X_2\right]}{\mathsf{P}\left[X_1 < 2X_2\right]} = \frac{1/15}{4/15} = \boxed{\frac{1}{4}}.
$$

(c) Find the marginal distribution of X_2, X_4.
Solution:

$$
\begin{aligned}
\int_0^3 \int_0^\infty \frac{4}{3} x_1 x_2^2 e^{-2x_3 - x_4}\, dx_3\, dx_1 &= \frac{4}{3} x_2^2 e^{-x_4}\left(\int_0^3 x_1\, dx_1\right)\left(\int_0^\infty e^{-2x_3}\, dx_3\right) \\
&= \frac{4}{3} x_2^2 e^{-x_4}\left(\frac{x_1^2}{2}\Big|_0^3\right)\left(-\frac{1}{2} e^{-2x_3}\Big|_0^\infty\right) \\
&= \frac{4}{3} x_2^2 e^{-x_4}\left(\frac{9}{2}\right)\left(\frac{1}{2}\right) \\
&= 3x_2^2 e^{-x_4}.
\end{aligned}
$$

The marginal distribution of X_2, X_4 is:

$$
\boxed{f(x_2, x_4) = \begin{cases} 3x_2^2 e^{-x_4}, & 0 < x_2 < 1,\ 0 < x_4 \\ 0, & \text{otherwise.} \end{cases}}
$$

(d) Find the marginal distribution of X_1, X_2, X_4.
Solution:

$$
\begin{aligned}
\int_0^\infty \frac{4}{3} x_1 x_2^2 e^{-2x_3 - x_4}\, dx_3 &= \frac{4}{3} x_1 x_2^2 e^{-x_4} \int_0^\infty e^{-2x_3}\, dx_3 = \frac{4}{3} x_1 x_2^2 e^{-x_4}\left(-\frac{1}{2} e^{-2x_3}\Big|_0^\infty\right) \\
&= \frac{4}{3} x_1 x_2^2 e^{-x_4}\left(\frac{1}{2}\right).
\end{aligned}
$$

The marginal distribution of X_1, X_2, X_4 is

$$
\boxed{f(x_1, x_2, x_4) = \begin{cases} \frac{2}{3} x_1 x_2^2 e^{-x_4}, & 0 < x_1 < 3,\ 0 < x_2 < 1,\ 0 < x_4 \\ 0, & \text{otherwise.} \end{cases}}
$$

Section 2.8

14. Let $X_1, \ldots, X_n$ be iid random variables with common mean μ and variance σ^2. Define $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$. Find $\mathbf{E}[\bar{X}]$ and $\mathbf{V}[\bar{X}]$.
Solution:

$$\begin{aligned}
\mathbf{E}\left[\bar{X}\right] &= \mathbf{E}\left[n^{-1} \sum_{i=1}^{n} X_i\right] = n^{-1}\,\mathbf{E}\left[\sum_{i=1}^{n} X_i\right] = n^{-1}\left(\mathbf{E}[X_1] + \cdots + \mathbf{E}[X_n]\right) \\
&= n^{-1}\Bigg(\underbrace{\mu + \cdots + \mu}_{n \text{ of them}}\Bigg) = n^{-1} \cdot n\mu = \boxed{\mu} \\
\mathbf{V}\left[\bar{X}\right] &= \mathbf{V}\left[n^{-1} \sum_{i=1}^{n} X_i\right] = \mathbf{V}\left[\sum_{i=1}^{n} n^{-1} X_i\right] = \sum_{i=1}^{n} \left(n^{-1}\right)^2 \mathbf{V}[X_i];\ \text{Corollary 2.8.2} \\
&= \sum_{i=1}^{n} n^{-2}\,\mathbf{V}[X_i] = n^{-2} \sum_{i=1}^{n} \mathbf{V}[X_i] = n^{-2}\left(\mathbf{V}[X_1] + \cdots + \mathbf{V}[X_n]\right) \\
&= n^{-2}\Bigg(\underbrace{\sigma^2 + \cdots + \sigma^2}_{n \text{ of them}}\Bigg) = n^{-2} \cdot n\sigma^2 = n^{-1}\sigma^2 = \boxed{\frac{\sigma^2}{n}}.
\end{aligned}$$

15. Let X and Y be random variables with $\mu_1 = 1, \mu_2 = 4, \sigma_1^2 = 4, \sigma_2^2 = 6, \rho = \frac{1}{2}$. Find the mean and variance of the random variable $Z = 3X - 2Y$.

Solution:

We should first find the covariance, $\mathbf{COV}(X, Y)$.

$$\rho = \frac{\mathbf{COV}(X,Y)}{\sigma_X \sigma_Y} \quad \Rightarrow \quad \frac{1}{2} = \frac{\mathbf{COV}(X,Y)}{\sqrt{4}\sqrt{6}} \quad \Rightarrow \quad \mathbf{COV}(X,Y) = \frac{\sqrt{4}\sqrt{6}}{2} = \sqrt{6}$$

$$\mathbf{E}[Z] = \mathbf{E}[3X - 2Y] = 3\,\mathbf{E}[X] - 2\,\mathbf{E}[Y] = 3\mu_1 - 2\mu_2 = 3(1) - 2(4) = \boxed{-5}$$

$$\begin{aligned}
\mathbf{V}[Z] &= \mathbf{V}[3X - 2Y] = 3^2\,\mathbf{V}[X] + (-2)^2\,\mathbf{V}[Y] + 2(3)(-2)\,\mathbf{COV}(X,Y) \\
&= 9\sigma_1^2 + 4\sigma_2^2 - 12\,\mathbf{COV}(X,Y) = 9(4) + 4(6) - 12\sqrt{6} \\
&= \boxed{60 - 12\sqrt{6} \approx 30.61}
\end{aligned}$$

16. Let X_1 and X_2 be independent random variables with nonzero variances. Find the covariance of $Y = X_1X_2$ and X_1 in terms of the means and variances of X_1 and X_2.

    Solution:

    Recall that $\mathbf{COV}(A, B) = \mathbf{E}\left[(A - \mu_A)(B - \mu_B)\right]$. In our case, we want to find $\mathbf{COV}(Y, X_1) = \mathbf{COV}(X_1X_2, X_1)$.

$$
\begin{aligned}
\mathbf{COV}(X_1X_2, X_1) &= \mathbf{E}\left[(X_1X_2 - \mathbf{E}[X_1X_2])(X_1 - \mathbf{E}[X_1])\right] \\
&= \mathbf{E}\left[(X_1X_2 - \mathbf{E}[X_1]\,\mathbf{E}[X_2])(X_1 - \mathbf{E}[X_1])\right];\ \text{since } X_1 \text{ and } X_2 \text{ are independent} \\
&= \mathbf{E}\left[(X_1X_2 - \mu_1\mu_2)(X_1 - \mu_1)\right] \\
&= \mathbf{E}\left[X_1^2X_2 - X_1X_2\mu_1 - \mu_1\mu_2X_1 + \mu_1^2\mu_2\right];\ \text{distribute} \\
&= \mathbf{E}\left[X_1^2X_2\right] - \mu_1\,\mathbf{E}\left[X_1X_2\right] - \mu_1\mu_2\,\mathbf{E}[X_1] + \mu_1^2\mu_2 \\
&= \mathbf{E}\left[X_1^2\right]\mathbf{E}[X_2] - \mu_1\,\mathbf{E}[X_1]\,\mathbf{E}[X_2] - \mu_1^2\mu_2 + \mu_1^2\mu_2;\ \text{by independence} \\
&= \left(\mathbf{V}[X_1] + \mathbf{E}[X_1]^2\right)\mu_2 - \mu_1^2\mu_2 \\
&= \left(\sigma_1^2 + \mu_1^2\right)\mu_2 - \mu_1^2\mu_2 \\
&= \sigma_1^2\mu_2 + \mu_1^2\mu_2 - \mu_1^2\mu_2 \\
&= \sigma_1^2\mu_2.
\end{aligned}
$$

Section 3.1

17. Consider a standard deck of 52 cards. Let X equal the number of aces in a sample of size 2.

    (a) If the sampling is with replacement, obtain the pmf of X.

    Solution:

    If sampling is with replacement (independent draws), then we have the binomial distribution. The probability of getting an ace is $4/52 = 1/13$. Let X represent the number of aces. The pmf of X is:

$$
p(x) = \begin{cases} \binom{2}{x}\left(\frac{1}{13}\right)^x\left(\frac{12}{13}\right)^{2-x}, & x = 0, 1, 2 \\ 0, & \text{otherwise} \end{cases}
$$

    (b) If the sampling is without replacement, obtain the pmf of X.

    Solution:

    If sampling is without replacement (draws depend on each other), we have the hypergeometric distribution. There is a possibility for 0, 1, or 2 aces in a sample of size 2. The pmf of X is:

$$
p(x) = \begin{cases} \frac{\binom{4}{x}\binom{48}{2-x}}{\binom{52}{2}}, & x = 0, 1, 2 \\ 0, & \text{otherwise} \end{cases}
$$

18. A traffic control engineer reports that 75% of the vehicles passing through a checkpoint are from within the state. What is the probability that fewer than 4 of the next 9 vehicles are from out of state? On average, how many cars will pass through the checkpoint? What is the variance?

Solution:

Let X be the number of out of state vehicles. This is a binomial distribution. $p = 0.25$; $n = 9$.

$$\mathsf{P}[X < 4] = \mathsf{P}[X \leq 3] = \sum_{k=0}^{3} \binom{9}{k} (0.25)^k (0.75)^{9-k} = 0.0751 + 0.2253 + 0.3003 + 0.2336 = \boxed{0.8343}.$$

Expected Value and Variance:

$$\begin{aligned} \mathbf{E}(X) &= np = 9(0.25) = \boxed{2.25} \\ \mathbf{V}(X) &= np(1-p) = 9(0.25)(0.75) = \boxed{1.6875}. \end{aligned}$$

19. Biologists doing studies in a particular environment often tag and release subjects in order to estimate the size of a population or the prevalence of certain features in the population. Ten animals of a certain population thought to be extinct (or near extinction) are caught, tagged, and released in a certain region. After a period of time, a random sample of 15 of this type of animal is selected in the region. What is the probability that 5 of those selected are tagged if there are 25 animals of this type in the region? On average, how many animals caught are tagged? What is the variance?

Solution:

Let X be the number of tagged animals selected. Use the hypergeometric distribution. $N = 25$; $n = 15$; $D = 10$; $x = 5$.

$$\mathsf{P}[X = 5] = \frac{\binom{10}{5}\binom{25-10}{15-5}}{\binom{25}{15}} = \frac{\binom{10}{5}\binom{15}{10}}{\binom{25}{15}} = \frac{(252)(3003)}{3268760} = \boxed{0.2315}.$$

Expected Value and Variance:

$$\begin{aligned} \mathbf{E}(X) &= \frac{nD}{N} = \frac{(15)(10)}{25} = \boxed{6} \\ \mathbf{V}(X) &= \frac{N-n}{N-1} \cdot n \cdot \frac{D}{N}\left(1 - \frac{D}{N}\right) \\ &= \left(\frac{25-15}{25-1}\right)(15)\left(\frac{10}{15}\right)\left(1 - \frac{10}{15}\right) \\ &= \left(\frac{5}{12}\right)(15)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = \boxed{\frac{25}{18} \approx 1.389}. \end{aligned}$$

20. What is the probability that a waitress will refuse to serve alcoholic beverages to only 2 minors if she randomly checks the IDs of 5 among 9 students, 4 of whom are minors? On average, how many minors will the waitress refuse to serve? What is the variance?

Solution:

Let X be the number of minors the waitress refuses to serve. Use the hypergeometric distribution. $x = 2$; $N = 9$; $n = 5$; $D = 4$.

$$\mathsf{P}[X = 2] = \frac{\binom{4}{2}\binom{9-4}{5-2}}{\binom{9}{5}} = \frac{\binom{4}{2}\binom{5}{3}}{\binom{9}{5}} = \frac{(6)(10)}{126} = \boxed{0.4762}.$$

Expected Value and Variance:

$$\begin{aligned}
\mathbf{E}(X) &= \frac{nD}{N} = \frac{(5)(4)}{9} = \boxed{2.22} \\
\mathbf{V}(X) &= \frac{N-n}{N-1} \cdot n \cdot \frac{D}{N}\left(1 - \frac{D}{N}\right) = \left(\frac{9-5}{9-1}\right)(5)\left(\frac{4}{9}\right)\left(1 - \frac{4}{9}\right) \\
&= \left(\frac{1}{2}\right)(5)\left(\frac{4}{9}\right)\left(\frac{5}{9}\right) = \boxed{\frac{50}{81} \approx 0.617}.
\end{aligned}$$

21. It is known that 60% of mice inoculated with a serum are protected form a certain disease. If 5 mice are inoculated, find the probability that

(a) none contracts the disease

(b) fewer than 2 contract the disease

(c) more than 3 contract the disease

Solution:

Let X be the number of mice who contract the disease. This is a binomial distribution. $n = 5$. $p = 0.4$.

(a) We want $\mathsf{P}[X = 0]$.

$$\mathsf{P}[X = 0] = \binom{5}{0}(0.4)^0(0.6)^{5-0} = \boxed{0.0778}.$$

(b) We want $\mathsf{P}[X < 2]$.

$$\mathsf{P}[X < 2] = \mathsf{P}[X \leq 1] = \sum_{k=0}^{1}\binom{5}{k}(0.4)^k(0.6)^{5-k} = \boxed{0.3370}.$$

(c) We want $\mathsf{P}[X > 3]$.

$$\begin{aligned}\mathsf{P}[X > 3] = 1 - \mathsf{P}[X \leq 3] &= 1 - \sum_{k=0}^{3} \binom{5}{k} (0.4)^k (0.6)^{5-k} \\ &= 1 - (0.07776 + 0.2592 + 0.3456 + 0.2304) = 1 - 0.91296 = \boxed{0.08704} \\ &\overset{OR}{=} \mathsf{P}[X \geq 4] = \sum_{k=4}^{5} \binom{5}{k} (0.4)^k (0.6)^{5-k} = 0.0768 + 0.01024 = \boxed{0.08704}.\end{aligned}$$

22. The probability that a person living in a certain city owns a cat is estimated to be 0.4. Find the probability that the tenth person randomly interviewed in that city is the third one to own a cat.

Solution:

Define Y to be the number of failures before the r^{th} success. We want 3 successes, so $r = 3$. The chance for success is $p = 0.4$. If we have 3 successes, then there must be $y = 7$ failures if we talk to 10 people.

$$\mathsf{P}[Y = 7] = \binom{7+3-1}{3-1} 0.4^3 (0.6)^7 = \binom{9}{2} (0.4)^3 (0.6)^7 = \boxed{0.0645}.$$

Using the Alternative Definition of Negative Binomial:
Let X be the number of interviews required for 3 people to own cats. Use the negative binomial distribution. $x = 10$; $k = 3$; $p = 0.4$.

$$\mathsf{P}[X = 10] = \binom{10-1}{3-1} (0.4)^3 (0.6)^{10-3} = \binom{9}{2} (0.4)^3 (0.6)^7 = \boxed{0.0645}.$$

23. It is known that 3% of people whose luggage is screened at an airport have questionable objects in their luggage. What is the probability that a string of 15 people pass through screening successfully before an individual is caught with a questionable object?

Solution:

Define Y to be the number of failures before the 1st success. If there the first one stopped is the 15th individual, then there were 14 failures. Find $\mathsf{P}[Y = 14]$.

$$\mathsf{P}[Y = 14] = (0.03)(0.97)^{14} = \boxed{0.0196}.$$

Using the Alternative Definition of Geometric:
Let X be the number of people screened until 1 person is caught. Use the geometric distribution. $p = 0.03$; $x = 15$; $k = 1$.

$$\mathsf{P}(X = 15) = (0.03)(0.97)^{15-1} = (0.03)(0.97)^{14} = \boxed{0.0196}.$$

Section 3.2

24. On average, 3 traffic accidents per month occur at a certain intersection. What is the probability that at any given month at this intersection

(a) exactly 5 accidents will occur?

(b) fewer than 3 accidents will occur?

(c) at least 2 accidents will occur?

Solution:

Let X be the number of accidents per month. Use the Poisson distribution. $\lambda = 3$; $w = 1$ month; $\lambda w = 3$. Use Table I from Appendix C of the textbook where needed.

(a) We want to find $\mathsf{P}[X = 5]$.

$$\mathsf{P}[X = 5] = \frac{e^{-3}3^5}{5!} = \boxed{0.1008}.$$

(b) We want to find $\mathsf{P}[X < 3]$.

$$\mathsf{P}[X < 3] = \mathsf{P}[X \leq 2] = \sum_{x=0}^{2} \frac{e^{-3}3^x}{x!} = \boxed{0.4232}.$$

(c) We want to find $\mathsf{P}[X \geq 2]$.

$$\mathsf{P}[X \geq 2] = 1 - \mathsf{P}[X < 2] = 1 - \mathsf{P}[X \leq 1] = 1 - \sum_{x=0}^{1} \frac{e^{-3}3^x}{x!} = 1 - 0.199 = \boxed{0.801}.$$

25. A certain area of the eastern United States is, on average, hit by 6 hurricanes a year. Find the probability that in 1.5 years that area will be hit by

(a) fewer than 4 hurricanes.

(b) anywhere from 6 to 8 hurricanes, inclusive.

Solution:

Let X be the number of hurricanes that hit the area. Use the Poisson distribution. $\lambda = 6$; $w = 1.5$ years; $\lambda w = 9$. Use Table I from Appendix C of the textbook where needed.

(a) We want to find $\mathsf{P}[X < 4]$

$$\mathsf{P}[X < 4] = \mathsf{P}[X \leq 3] = \sum_{x=0}^{3} \frac{e^{-9}9^x}{x!} = \boxed{0.021}.$$

(b) We want to find $\mathsf{P}[6 \le X \le 8]$.

$$\mathsf{P}(6 \le X \le 8) = \sum_{x=6}^{8} \frac{e^{-9}9^x}{x!} = \sum_{i=x}^{8} \frac{e^{-9}9^x}{x!} - \sum_{x=0}^{5} \frac{e^{-9}9^x}{x!} = \mathsf{P}(X \le 8) - \mathsf{P}(X \le 5)$$
$$= 0.456 - 0.116 = \boxed{0.34}.$$

26. On the average, a grocer sells three of a certain article per week. How many of these should he have in stock so that the chance of his running out within a week is less than 0.01? Assume a Poisson Distribution.

Solution:

Define X to be the number of items sold during the week. Using the Poisson Distribution, we have $\lambda = 3$, $w = 1$, so $\lambda w = 3$.

To help us figure out how to solve the problem, let us think about what it is asking. If 0 items are sold, then we didn't run out. If 1 item is sold, there is a possibility that we run out. If 2 items are sold, there is a larger possibility that we run out. It all depends on how many the grocer has in stock.

Let k be the number that the grocer has in stock. If there are requests for $k+1$, $k+2$, etc of these articles, then the grocer won't have enough (run out). If the grocer has a request for $\le k$ of these items, then the grocer has enough (not run out). We want the probability that we run out to be less than 0.01, or $\mathsf{P}[X > k] < 0.01$.

$$\begin{aligned}\mathsf{P}[X > k] < 0.01 &\Rightarrow -\mathsf{P}[X > k] > -0.01\\ &\Rightarrow 1 - \mathsf{P}[X > k] > 1 - 0.01\\ &\Rightarrow \mathsf{P}[X \le k] > 0.99\\ &\Rightarrow \sum_{i=0}^{k} \frac{e^{-3}3^i}{i!} > 0.99\end{aligned}$$

Using the Poisson Table, if $k = 7$, then $\mathsf{P}[X \le 7] = 0.988 \not> 0.99$. If $k = 8$, then $\mathsf{P}[X \le 8] = 0.996 > 0.99$.

The grocer should have 8 in stock.

27. Find moment generating functions for the following probability distributions.

(a) Let X be a random variable and n a positive integer. Let $0 < p < 1$. The pmf of X is given by

$$p(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x = 0, 1, 2, \ldots, n \\ 0, & \text{otherwise.} \end{cases}$$

Solution:

$$\begin{aligned} M_X(t) = \mathbf{E}\left(e^{tX}\right) &= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=0}^{n} \binom{n}{x} \left(e^t\right)^x p^x (1-p)^{n-x} \\ &= \sum_{x=0}^{n} \binom{n}{x} \left(pe^t\right)^x (1-p)^{n-x} \\ &= \boxed{\left(pe^t + 1 - p\right)^n, \quad -\infty < t < \infty}; \text{ by Binomial Theorem: } \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n. \end{aligned}$$

(b) Let X be a random variable and $\lambda > 0$ be a constant. The pmf of X is given by

$$p(x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!}, & x = 0, 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

Solution:

$$\begin{aligned} M_X(t) = \mathbf{E}\left(e^{tX}\right) &= \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\left(\lambda e^t\right)^x}{x!} \\ &= e^{-\lambda} e^{\lambda e^t}; \text{ by Power Series } e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!} \\ &= e^{-\lambda + \lambda e^t} \\ &= \boxed{e^{\lambda\left(e^t - 1\right)}, \quad -\infty < t < \infty}. \end{aligned}$$

(c) Let X be a random variable and $0 < p < 1$. The pmf of X is given by

$$p(x) = \begin{cases} (1-p)^{x-1}p, & x = 1, 2, 3, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

Solution:

$$\begin{aligned}
M_X(t) = \mathbf{E}\left(e^{tX}\right) &= \sum_{x=1}^{\infty} e^{tx}(1-p)^{x-1}p \\
&= p\sum_{x=1}^{\infty} e^{tx}(1-p)^{x}(1-p)^{-1} \\
&= \frac{p}{1-p}\sum_{x=1}^{\infty} \left(e^{t}\right)^{x}(1-p)^{x} \\
&= \frac{p}{1-p}\sum_{x=1}^{\infty} \left[(1-p)e^{t}\right]^{x} \\
&= \frac{p}{1-p}\left\{\sum_{x=0}^{\infty} \left[(1-p)e^{t}\right]^{x} - \left[(1-p)e^{t}\right]^{0}\right\} \\
&= \frac{p}{1-p}\left\{\frac{1}{1-(1-p)e^{t}} - 1\right\}; \text{ Geometric Series and } \left|(1-p)e^{t}\right| < 1 \\
&= \frac{p}{1-p}\left\{\frac{1}{1-(1-p)e^{t}} - \frac{1-(1-p)e^{t}}{1-(1-p)e^{t}}\right\}; \;\; (1-p)e^{t} < 1 \\
&= \frac{p}{1-p}\left\{\frac{1-1+(1-p)e^{t}}{1-(1-p)e^{t}}\right\}; \;\; e^{t} < (1-p)^{-1} \\
&= \frac{p}{1-p}\left\{\frac{(1-p)e^{t}}{1-(1-p)e^{t}}\right\}; \;\; t < \ln\left[(1-p)^{-1}\right] \\
&= \boxed{\frac{pe^{t}}{1-(1-p)e^{t}}; \;\; t < -\ln(1-p)}.
\end{aligned}$$

(d) Let X be a random variable and $a < b$ be constants. The pdf of X is given by

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise.} \end{cases}$$

Solution:

$$\begin{aligned}
M_X(t) = \mathbf{E}\left(e^{tX}\right) &= \int_a^b e^{tx} \cdot \frac{1}{b-a}\,dx = \frac{1}{b-a} \cdot \frac{1}{t}e^{tx}\Big|_a^b = \frac{1}{t(b-a)}\left(e^{tb} - e^{ta}\right) \\
&= \boxed{\frac{e^{tb} - e^{ta}}{t(b-a)}, \;\; t \neq 0}.
\end{aligned}$$

(e) Let X be a random variable, $-\infty < \mu < \infty$ a constant, and $\sigma^2 > 0$ a constant. The pdf of X is given by

$$f(x) = \begin{cases} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, & -\infty < x < \infty \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned}
M_X(t) = \mathbf{E}\left(e^{tX}\right) &= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x^2-2\mu x+\mu^2-2\sigma^2 tx)}\, dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\left[x^2-2x\left(\mu+\sigma^2 t\right)+\mu^2\right]}\, dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\left[x^2-2x\left(\mu+\sigma^2 t\right)+\mu^2+(2\mu\sigma^2 t+(\sigma^2)^2t^2)-(2\mu\sigma^2 t+(\sigma^2)^2 t)\right]}\, dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\left[x-\left(\mu+\sigma^2 t\right)\right]^2+\frac{2\mu\sigma^2 t+(\sigma^2)^2 t}{2\sigma^2}}\, dx \\
&= e^{\mu t+\frac{\sigma^2 t}{2}} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\left[x-\left(\mu+\sigma^2 t\right)\right]^2}\, dx}_{\text{Integrates to 1; Just shifted the original}} \\
&= \boxed{e^{\mu t+\frac{\sigma^2 t}{2}}, \quad -\infty < t < \infty}.
\end{aligned}$$

(f) Let X be a random variable, $\alpha > 0$ a constant, and $\theta > 0$ a constant. Let $\Gamma(\alpha)$ be a Gamma Function evaluated at α. The pdf of X is given by

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}, & 0 \le x < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Solution:

$$\begin{aligned}
M_X(t) = \mathbf{E}\left(e^{tX}\right) &= \int_0^{\infty} e^{tx} \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}\, dx \\
&= \int_0^{\infty} \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x\left(\theta^{-1}-t\right)}\, dx; \text{ Let } u = x\left(\theta^{-1}-t\right) \\
&= \int_0^{\infty} \frac{1}{\Gamma(\alpha)\theta^\alpha} \left(\frac{u}{\theta^{-1}-t}\right)^{\alpha-1} e^{-u} \frac{du}{\theta^{-1}-t} \\
&= \left(\frac{1}{\theta^{-1}-t}\right)^{\alpha-1} \cdot \frac{1}{\theta^{-1}-t} \cdot \frac{1}{\theta^\alpha} \cdot \underbrace{\int_0^{\infty} \frac{1}{\Gamma(\alpha)1^\alpha} u^{\alpha-1} e^{-u}\, du}_{\text{Special Case of Original with } \theta = 1} \\
&= \frac{1}{\left(\frac{1}{\theta}-t\right)^\alpha} \cdot \frac{1}{\theta^\alpha}(1) = \frac{1}{\left[\theta\left(\frac{1}{\theta}-t\right)\right]^\alpha} \\
&= \boxed{\frac{1}{(1-\theta t)^\alpha}, \quad t < \frac{1}{\theta}}.
\end{aligned}$$

(g) Let X be a random variable. The pdf of X is given by

$$f(x) = \begin{cases} \frac{4}{255}x^3, & -1 < x < 4 \\ 0, & \text{otherwise.} \end{cases}$$

Solution:

$$\begin{aligned}
M_X(t) &= \mathbf{E}\left(e^{tX}\right) = \int_{-1}^{4} e^{tx} \cdot \frac{4}{255}x^3\,dx = \frac{4}{255}\int_{-1}^{4} x^3 e^{tx}\,dx \\
&= \frac{4}{255}\left[x^3\frac{1}{t}e^{tx} - 3x^2\frac{1}{t^2}e^{tx} + 6x\frac{1}{t^3}e^{tx} - 6\frac{1}{t^4}e^{tx}\right]\Bigg|_{-1}^{4} \\
&= \frac{4}{255}\cdot\frac{1}{t}e^{tx}\left(x^3 - 3x^2\frac{1}{t} + 6x\frac{1}{t^2} - 6\frac{1}{t^3}\right)\Bigg|_{-1}^{4} \\
&= \frac{4}{255}\cdot\frac{1}{t}e^{4t}\left(4^3 - 3(4)^2\frac{1}{t} + 6(4)\frac{1}{t^2} - 6\frac{1}{t^3}\right) \\
&\quad - \frac{4}{255}\cdot\frac{1}{t}e^{-t}\left((-1)^3 - 3(-1)^2\frac{1}{t} + 6(-1)\frac{1}{t^2} - 6\frac{1}{t^3}\right) \\
&= \frac{4}{255}\cdot\frac{1}{t}e^{4t}\left(64 - \frac{48}{t} + \frac{24}{t^2} - \frac{6}{t^3}\right) - \frac{4}{255}\cdot\frac{1}{t}e^{-t}\left(-1 - \frac{3}{t} - \frac{6}{t^2} - \frac{6}{t^3}\right) \\
&= \boxed{\frac{4}{255}\cdot\frac{1}{t}e^{4t}\left(64 - \frac{48}{t} + \frac{24}{t^2} - \frac{6}{t^3}\right) + \frac{4}{255}\cdot\frac{1}{t}e^{-t}\left(1 + \frac{3}{t} + \frac{6}{t^2} + \frac{6}{t^3}\right),\quad t \neq 0}.
\end{aligned}$$

28. Let X_1 and X_2 be independent random variables. The pdf of X_1 is

$$f_1(x_1) = \begin{cases} \frac{1}{\Gamma(2)\left(\frac{1}{2}\right)^2} x e^{-2x}, & 0 \leq x_1 < \infty \\ 0, & \text{otherwise.} \end{cases}$$

The pdf of X_2 is

$$f_2(x_2) = \begin{cases} \frac{1}{\Gamma(4)\left(\frac{1}{2}\right)^4} x^3 e^{-2x}, & 0 \leq x_2 < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Find the distribution of $Y = X_1 + X_2$ using MGFs.

Solution:

The pdfs given for X_1 and X_2 are examples of the pdfs in Question 27f. From this part, we know that the MGF for X_1 and the MGF for X_2 are:

$$M_{X_1}(t) = \frac{1}{\left(1 - \frac{1}{2}t\right)^2}, \quad t < 2$$

and

$$M_{X_2}(t) = \frac{1}{\left(1 - \frac{1}{2}t\right)^4}, \quad t < 2.$$

First, find $M_Y(t)$.

$$M_Y(t) = \mathbf{E}\left[e^{tY}\right] = \mathbf{E}\left[e^{t(X_1+X_2)}\right] = \mathbf{E}\left[e^{tX_1}e^{tX_2}\right] = \mathbf{E}\left[e^{tX_1}\right]\mathbf{E}\left[e^{tX_2}\right]; \text{ by Independence}$$

$$= M_{X_1}(t)M_{X_2}(t) = \frac{1}{\left(1 - \frac{1}{2}t\right)^2} \cdot \frac{1}{\left(1 - \frac{1}{2}t\right)^4} = \boxed{\frac{1}{\left(1 - \frac{1}{2}t\right)^6}, \quad t < 2}.$$

This matches the pdf in Question 27f, with $\theta = \frac{1}{2}$ and $\alpha = 6$. Therefore, Y has the pdf

$$\boxed{f_Y(y) = \begin{cases} \frac{1}{\Gamma(6)\left(\frac{1}{2}\right)^6} y^5 e^{-2y}, & 0 \leq y \leq \infty \\ 0, & \text{otherwise.} \end{cases}}$$

29. Let X_1 and X_2 be independent random variables, such that

$$p_1(x_1) = \begin{cases} \left(\frac{9}{10}\right)^{x-1}\left(\frac{1}{10}\right), & x_1 = 1, 2, 3, \ldots \\ 0, & \text{otherwise} \end{cases}$$

and

$$p_2(x_2) = \begin{cases} \left(\frac{3}{10}\right)^{x-1}\left(\frac{7}{10}\right), & x_2 = 1, 2, 3, \ldots \\ 0, & \text{otherwise} \end{cases}.$$

Use MGFs to find the pdf of $Y = X_1 + X_2$.

Solution:

The pmf's given for X_1 and X_2 are examples of the pmfs in Question 27c. From this part, we know that the MGF for X_1 and the MGF for X_2 are:

$$M_{X_1}(t) = \frac{\left(\frac{1}{10}\right)e^t}{1 - \frac{9}{10}e^t}, \quad t < -\ln\left(\frac{9}{10}\right)$$

and

$$M_{X_2}(t) = \frac{\frac{7}{10}e^t}{1 - \frac{3}{10}e^t}, \quad t < -\ln\left(\frac{3}{10}\right).$$

We want to find $M_Y(t)$.

$$\begin{aligned} M_Y(t) &= \mathbf{E}\left[e^{tY}\right] = \mathbf{E}\left[e^{t(X_1+X_2)}\right] = \mathbf{E}\left[e^{tX_1}e^{tX_2}\right] = \mathbf{E}\left[e^{tX_1}\right]\mathbf{E}\left[e^{tX_2}\right]; \text{ by Independence} \\ &= M_{X_1}(t)M_{X_2}(t) = \frac{\left(\frac{1}{10}\right)e^t}{1 - \frac{9}{10}e^t} \cdot \frac{\frac{7}{10}e^t}{1 - \frac{3}{10}e^t} \\ &= \boxed{\frac{\frac{7}{100}e^{2t}}{\left(1 - \frac{9}{10}e^t\right)\left(1 - \frac{3}{10}e^t\right)}, \quad t < -\ln\left(\frac{9}{10}\right)}. \end{aligned}$$

30. Suppose X_1 and X_2 are random variables such that their joint pdf is

$$f(x_1, x_2) = \begin{cases} x_1 e^{-x_2}, & 0 < x_1 < x_2 < \infty \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the moment generating function of X_1 and X_2, $M(t_1, t_2)$.

Solution:

Keep in mind that we *do not* know if X_1 and X_2 are independent. Because of this, we cannot use any of the independence rules.

$$
\begin{aligned}
M(t_1,t_2) &= \mathbf{E}\left[e^{t_1X_1+t_2X_2}\right] = \mathbf{E}\left[e^{t_1X_1}e^{t_2X_2}\right] = \int_0^\infty \int_0^{x_2} e^{t_1x_1}e^{t_2x_2}x_1e^{-x_2}\,dx_1\,dx_2 \\
&= \int_0^\infty e^{-x_2+t_2x_2}\left[-\frac{x_1}{t_1}e^{-t_1x_1}-\frac{1}{t_1^2}e^{-t_1x_1}\Big|_0^{x_2}\right]dx_2 \\
&= \int_0^\infty e^{-x_2+t_2x_2}\left[-\frac{x_2}{t_1}e^{-t_1x_2}-\frac{1}{t_1^2}e^{-t_1x_2}+\frac{1}{t_1^2}\right]dx_2 \\
&= \int_0^\infty \left(-\frac{1}{t_1}x_2e^{-x_2+t_2x_2-t_1x_2}-\frac{1}{t_1^2}e^{-x_2+t_2x_2-t_1x_2}+\frac{1}{t_1^2}e^{-x_2+t_2x_2}\right)dx_2 \\
&= \int_0^\infty \left(-t_1^{-1}x_2e^{-x_2(1+t_1-t_2)}-t_1^{-2}e^{-x_2(1+t_1-t_2)}+t_1^{-2}e^{-x_2(1-t_2)}\right)dx_2 \\
&= -t_1^{-1}\left[-\frac{x_2}{1+t_1-t_2}e^{-(1+t_1-t_2)x_2}-\frac{1}{(1+t_1-t_2)^2}e^{-(1+t_1-t_2)x_2}\Big|_0^\infty\right] \\
&\qquad - t_1^{-2}\left[\frac{-1}{1+t_1-t_2}e^{-x_2(1+t_1-t_2)}\Big|_0^\infty\right]+t_1^{-2}\left[\frac{-1}{1-t_2}e^{-x_2(1-t_2)}\Big|_0^\infty\right] \\
&= -t_1^{-1}\left[-0--0-\left(-0-\frac{1}{1+t_1-t_2}\right)\right]-t_1^{-2}\left[0-\frac{-1}{1+t_1-t_2}\right]+t_1^{-2}\left[0-\frac{-1}{1-t_2}\right] \\
&= -t_1^{-1}\left[\frac{1}{1+t_1-t_2}\right]-t_1^{-2}\left[\frac{1}{1+t_1-t_2}\right]+t_1^{-2}\left[\frac{1}{1-t_2}\right] \\
&= \frac{-1}{1+t_1-t_2}\left(\frac{1}{t_1}+\frac{1}{t_1^2}\right)+\frac{1}{t_1^2(1-t_2)} = \frac{-1}{1+t_1-t_2}\cdot\frac{1}{t_1^2}(t_1+1)+\frac{1}{t_1^2(1-t_2)} \\
&= \frac{1}{t_1^2}\left[\frac{-(t_1+1)}{1+t_1-t_2}+\frac{1}{1-t_2}\right] = \frac{1}{t_1^2}\left[\frac{1}{1-t_2}-\frac{1+t_1}{1+t_1-t_2}\right] \\
&= \frac{1}{t_1^2}\left[\frac{1+t_1-t_2-(1+t_1)(1-t_2)}{(1-t_2)(1+t_1-t_2)}\right] = \frac{1}{t_1^2}\left[\frac{1+t_1-t_2-(1-t_2+t_1-t_1t_2)}{(1-t_2)(1+t_1-t_2)}\right] \\
&= \frac{1}{t_1^2}\left[\frac{t_1t_2}{(1-t_2)(1+t_1-t_2)}\right] \\
&= \boxed{\frac{t_2}{t_1(1-t_2)(1+t_1-t_2)},\quad t_1\neq 0; t_2\neq 1; 1+t_1-t_2\neq 0}.
\end{aligned}
$$

(b) Find the marginal distributions of X_1 and X_2.

Solution:

Marginal Distribution of X_1:

$$\int_{x_1}^{\infty} x_1 e^{-x_2}\, dx_2 = -x_1 e^{-x_2}\Big|_{x_1}^{\infty} \\ = x_1 e^{-x_1}.$$

$$\boxed{f(x_1) = \begin{cases} x_1 e^{-x_1}, & 0 < x_1 < \infty \\ 0, & \text{otherwise.} \end{cases}}$$

Marginal Distribution of X_2:

$$\int_{0}^{x_2} x_1 e^{-x_2}\, dx_1 = \frac{x_1^2}{2} \cdot e^{-x_2}\Big|_{0}^{x_2} \\ = \frac{x_2^2}{2} \cdot e^{-x_2}$$

$$\boxed{f(x_2) = \begin{cases} \frac{1}{2} x_2^2 e^{-x_2}, & 0 < x_2 < \infty \\ 0, & \text{otherwise.} \end{cases}}$$

(c) Find the moment generating function of X_1.

Solution:

$$M_{X_1}(t) = \mathbf{E}\left[e^{tX_1}\right] = \int_0^{\infty} e^{tx_1} x_1 e^{-x_1}\, dx_1 = \int_0^{\infty} x_1 e^{-x_1 + tx_1}\, dx_1 = \int_0^{\infty} x_1 e^{-x_1(1-t)}\, dx_1$$

$$= -\frac{x_1}{1-t} e^{-(1-t)x_1} - \frac{1}{(1-t)^2} e^{-(1-t)x_1}\Big|_0^{\infty} = 0 - 0 - \left(-0 - \frac{1}{(1-t)^2} e^0\right)$$

$$= \boxed{\frac{1}{(1-t)^2}, \quad t \neq 1}.$$

(d) Find the moment generating function of X_2.
Solution:

$$\begin{aligned}
M_{X_2}(t) = \mathbf{E}\left[e^{tX_2}\right] &= \int_0^\infty e^{tx_2} \cdot \frac{1}{2} x_2^2 e^{-x_2}\, dx_2 = \frac{1}{2}\int_0^\infty x_2^2 e^{-x_2+tx_2}\, dx_2 \\
&= \frac{1}{2}\int_0^\infty x_2^2 e^{-x_2(1-t)}\, dx_2 \\
&= \frac{1}{2}\left[-\frac{x_2^2}{1-t}e^{-(1-t)x_2} - \frac{2x_2}{(1-t)^2}e^{-(1-t)x_2} - \frac{2}{(1-t)^3}e^{-(1-t)x_2}\Bigg|_0^\infty\right] \\
&= \frac{1}{2}\left[-0-0-0-\left(-0-0-\frac{2}{(1-t)^3}e^0\right)\right] \\
&= \frac{1}{2}\left[\frac{2}{(1-t)^3}\right] = \boxed{\frac{1}{(1-t)^3},\ \ t \neq 1}.
\end{aligned}$$