Notes:

- THIS STUDY GUIDE COVERS SECTIONS 3.3–3.7; 5.1–5.3
- You should also study all of your old homework assignments and in-class notes. Possible exam questions may come from those as well. This study guide is NOT exhaustive.
- You should also review material from the entire semester (not just the material presented here). A better summary of old material will come closer to the final exam date.
- REMINDERS: No cheat sheet. You may use a scientific, *but not graphing* calculator.

Section 3.3: Gamma, Chi-Square, and Beta Distributions

1. If X is $\chi^2(5)$, determine the constants c and d so that $\mathsf{P}[c < X < d] = 0.95$ and $\mathsf{P}[X < c] = 0.025$.

2. Find $\mathsf{P}[3.28 < X < 25.2]$ if X has a gamma distribution with $\alpha = 3$ and $\beta = 4$. Hint: Consider the probability of the equivalent event $1.64 < Y < 12.6$, where $Y = 2X/4 = X/2$.

3. Let X_1, X_2, and X_3 be iid random variables, each with pdf $f(x) = e^{-x}$, $0 < x < \infty$, zero elsewhere.
   (a) Find the distribution of $Y = \min(X_1, X_2, X_3)$.
   (b) Find the distribution of $Y = \max(X_1, X_2, X_3)$.

4. Determine the constant c so that $f(x)$ is a β pdf:

$$f(x) = \begin{cases} cx^4(1-x)^5, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Section 3.4: Normal Distribution

5. State the MGF of a random variable $X \sim \mathsf{N}(\mu, \sigma^2)$.

6. Find the value of z_p where $p = 0.95$.

7. Find the value of z_p where $p = 0.9207$.

8. If X has the MGF
$$M_X(t) = e^{4t+64t^2},$$
what distribution does X have and what are its parameter values?

9. Suppose $X \sim \mathsf{N}(100, 16)$. Find the value of z for:
   (a) $x = 90$
   (b) $x = 110$
   (c) $x = 80$
   (d) $x = 105$

10. Suppose $X \sim \mathsf{N}(100, 16)$. Find the following probabilities.
   (a) $\mathsf{P}[X < 90]$.
   (b) $\mathsf{P}[105 < X < 110]$
   (c) $\mathsf{P}[X \geq 90]$.
   (d) $\mathsf{P}[90 < X \leq 105]$

11. Suppose $X \sim \mathsf{N}(100, 16)$.
   (a) Is a value of 90 or smaller likely to occur? Why or why not?
   (b) Is a value of 80 or smaller likely to occur? Why or why not?

12. If the random variable $X \sim \mathsf{N}(\mu, \sigma^2)$, where $\sigma^2 > 0$, then show that the random variable $(X - \mu)^2/\sigma^2 \sim \chi^2(1)$.

13. Remember the following corollary:

   Corollary 1. *Let $X_1, \ldots, X_n$ be iid random variables with common $\mathsf{N}(\mu, \sigma^2)$ distribution. Let $\bar{X} = n^{-1}\sum_{i=1}^{n} X_i$. Then $\bar{X} \sim \mathsf{N}(\mu, \sigma^2/n)$.*

Section 3.5: Multivariate Normal Distribution

14. Let X and Y have bivariate normal distribution with parameters $\mu_1 = 3$, $\mu_2 = 1$, $\sigma_1^2 = 16$, $\sigma_2^2 = 25$, and $\rho = 3/5$. Determine the following probabilities.

    (a) $\mathsf{P}[3 < Y < 8]$

    (b) $\mathsf{P}[3 < Y < 8 \mid X = 7]$

    (c) $\mathsf{P}[-3 < X < 3]$

    (d) $\mathsf{P}[-3 < X < 3 \mid Y = -4]$

15. Let X and Y have bivariate normal distribution with parameters $\mu_1 = 5$, $\mu_2 = 10$, $\sigma_1^2 = 1$, $\sigma_2^2 = 25$, and $\rho > 0$. if $\mathsf{P}[4 < Y < 16 \mid X = 5] = 0.954$, determine ρ.

Section 3.6: t- and F- distributions

16. Let T have a t-distribution with 14 degrees of freedom. Determine b so that $\mathsf{P}[-b < T < b] = 0.90$.

17. Find the corresponding t-values or areas.

    (a) Find the t-value such that $\mathsf{P}\left(T > t_{0.01}(16)\right) = 0.01$.

    (b) Find the value of $t_{0.975}(14)$.

    (c) Find $\mathsf{P}\left(-t_{0.025}(v) < T < t_{0.05}(v)\right)$. v is unknown.

    (d) Find k such that $\mathsf{P}(T > k) = 0.025$ for 23 degrees of freedom.

Section 3.7: Mixture Distributions

18. Suppose you have the mixture $0.75\mathsf{N}(0, 1) + 0.25\mathsf{N}(1.5, 4)$.

    (a) Find its expected value.

    (b) Find its variance.

19. Suppose you have the mixture $0.5\mathsf{N}(-1, 1) + 0.5\mathsf{N}(1, 1)$.

    (a) Find its expected value.

    (b) Find its variance.

20. Suppose you have the mixture $0.25\mathsf{Pois}(5) + 0.75\chi^2(8)$.

    (a) Find its expected value.

    (b) Find its variance.

Section 5.1: Convergence in Probability

21. Suppose $X_1, \dots, X_n$ is a random sample from a Uniform$(0, \theta)$ distribution. Suppose θ is unknown. An intuitive estimate of θ is the maximum of the sample. Let $Y_n = \max\{X_1, \dots, X_n\}$. Note: A uniform random variable $X \sim \mathsf{Unif}(a, b)$ has the pdf

$$f(x) = \frac{1}{b-a}, \quad -\infty < a < x < b < \infty.$$

   (a) Show that the CDF of Y_n is

$$F_{Y_n}(t) = \begin{cases} 1, & t > \theta \\ \left(\frac{t}{\theta}\right)^n, & 0 < t \leq \theta \\ 0, & t \leq 0. \end{cases}$$

   (b) Find the PDF of Y_n.

   (c) Show that Y_n is a biased estimator of θ.

   (d) Show that $\frac{n+1}{n} Y_n$ is an unbiased estimator of θ.

   (e) Show that $Y_n \xrightarrow{P} \theta$, i.e. show that Y_n is a consistent estimator of θ.

   (f) Show that $\frac{n+1}{n} Y_n$ is a consistent estimator of θ.

22. Suppose $X_1, \dots, X_n$ is a random sample from a Uniform$(0, \theta)$ distribution. Suppose θ is unknown. Show that $\bar{X}_n$ is a consistent estimator of $\theta/2$.

Section 5.2: Convergence in Distribution

23. Suppose $X_1, \dots, X_n$ is a random sample from a Uniform$(0, \theta)$ distribution. Suppose θ is unknown. An intuitive estimate of θ is the maximum of the sample. Let $Y_n = \max\{X_1, \dots, X_n\}$. Consider the random variable $Z_n = n(\theta - Y_n)$. Let $t \in (0, n\theta)$. Show that $Z_n \xrightarrow{D} Z$, where $Z \sim \mathsf{Exp}(\theta)$.

24. Let $Z_n \sim \chi^2(n)$. Find the limiting distribution of the random variable $Y_n = (Z_n - n)/\sqrt{2n}$ by using Moment Generating Functions and Taylor's Expansion.

Section 5.3: Central Limit Theorem

25. Let $\bar{X}$ denote the mean of a random sample of size 128 from a Gamma Distribution with $\alpha = 2$ and $\beta = 4$. Approximate $\mathsf{P}[7 < \bar{X} < 9]$.

26. Let $Y \sim \mathsf{Bin}\left(72, \frac{1}{3}\right)$. Approximate $\mathsf{P}[22 \leq Y \leq 28]$.

27. Let $Y \sim \mathsf{Bin}\left(400, \frac{1}{5}\right)$. Compute an approximate value of $\mathsf{P}\left[0.25 < \frac{Y}{400}\right]$.

28. If $Y \sim \mathsf{Bin}\left(100, \frac{1}{2}\right)$, approximate the value of $\mathsf{P}[Y = 50]$.

Solutions

Section 3.3

1. If X is $\chi^2(5)$, determine the constants c and d so that $\mathsf{P}[c < X < d] = 0.95$ and $\mathsf{P}[X < c] = 0.025$.

   Solution:

   We can use Table II from the back of the textbook to help identify the values of c and d. In this scenario, there are 5 degrees of freedom.

   $$\mathsf{P}[X < c] = 0.025 \Rightarrow \boxed{c = 0.831}.$$

   We can now identify d:

   $$\begin{aligned} 0.95 = \mathsf{P}[c < X < d] = \mathsf{P}[0.831 < X < d] = \mathsf{P}[X < d] - \mathsf{P}[X < 0.831] &= \mathsf{P}[X < d] - 0.025 \\ \Rightarrow 0.975 &= \mathsf{P}[X < d] \\ \Rightarrow &\boxed{d = 12.833}. \end{aligned}$$

2. Find $\mathsf{P}[3.28 < X < 25.2]$ if X has a gamma distribution with $\alpha = 3$ and $\beta = 4$. Hint: Consider the probability of the equivalent event $1.64 < Y < 12.6$, where $Y = 2X/4 = X/2$.

   Solution:

   If we use the hint, we need to identify the distribution of $Y = X/2$.

   $$M_X(t) = \frac{1}{(1-4t)^3}, \quad t < \frac{1}{4}$$
   $$M_Y(t) = \mathbf{E}[e^{tY}] = \mathbf{E}[e^{tX/2}] = \mathbf{E}[e^{(t/2)X}] = M_X(t/2) = \frac{1}{\left(1 - 4\left(\frac{t}{2}\right)\right)^3} = \frac{1}{(1-2t)^3}, \quad t < \frac{1}{2}.$$

   This is the MGF of a χ^2 random variable with degrees of freedom of $r/2 = 3 \Rightarrow r = 6$.

   $$\mathsf{P}[3.28 < X < 25.2] = \mathsf{P}[1.64 < Y < 12.6] = \mathsf{P}[Y < 12.6] - \mathsf{P}[Y < 1.64] \approx 0.95 - 0.05 = \boxed{0.90}.$$

3. Let X_1, X_2, and X_3 be iid random variables, each with pdf $f(x) = e^{-x}, \ 0 < x < \infty$, zero elsewhere.

   (a) Find the distribution of $Y = \min(X_1, X_2, X_3)$.

   Solution:

   First, we find the CDF of X.

   $$\mathsf{P}[X \le x] = \int_0^x e^{-t}\,dt = -e^{-t}\Big|_0^x = -e^{-x} + e^0 = 1 - e^{-x}.$$

   The CDF of X is:

   $$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x} & 0 \le x < \infty \end{cases}$$

We can find the CDF of Y:

$$\begin{aligned}
\mathsf{P}[Y<y] &= 1-\mathsf{P}[Y>y] \\
&= 1-\mathsf{P}[X_1>y, X_2>y, X_3>y] \\
&= 1-\mathsf{P}[X_1>y]\mathsf{P}[X_2>y]\mathsf{P}[X_3>y]; \text{ by independence} \\
&= 1-(\mathsf{P}[X_1>y])^3; \text{ since the variables are iid} \\
&= 1-(1-\mathsf{P}[X_1<y])^3 \\
&= 1-\left(1-\left[1-e^{-y}\right]\right)^3 \\
&= 1-\left(e^{-y}\right)^3 \\
&= 1-e^{-3y}.
\end{aligned}$$

The CDF of Y is:

$$\boxed{F_Y(y)=\begin{cases}0, & y<0\\ 1-e^{-3y}, & 0\le y<\infty\end{cases}}$$

The PDF of Y is:

$$\boxed{f_Y(y)=\begin{cases}3e^{-3y}, & 0<y<\infty\\ 0, & \text{otherwise}\end{cases}}$$

(b) Find the distribution of $Y=\max(X_1,X_2,X_3)$.

Solution:

Recall that the CDF of X is:

$$F_X(x)=\begin{cases}0, & x<0\\ 1-e^{-x} & 0\le x<\infty\end{cases}$$

We can find the CDF of Y:

$$\begin{aligned}
\mathsf{P}[Y<y] &= \mathsf{P}[X_1<y, X_2<y, X_3<y] \\
&= \mathsf{P}[X_1<y]\mathsf{P}[X_2<y]\mathsf{P}[X_3<y]; \text{ by independence} \\
&= (\mathsf{P}[X_1<y])^3; \text{ since the variables are iid} \\
&= \left(1-e^{-y}\right)^3.
\end{aligned}$$

The CDF of Y is:

$$\boxed{F_Y(y)=\begin{cases}0, & y<0\\ (1-e^{-y})^3 & 0\le y<\infty\end{cases}}$$

The PDF of Y is:

$$\boxed{f_Y(y)=\begin{cases}3e^{-y}\left(1-e^{-y}\right)^2 & 0<y<\infty\\ 0, & \text{otherwise.}\end{cases}}$$

4. Determine the constant c so that $f(x)$ is a β pdf:

$$f(x) = \begin{cases} cx^4(1-x)^5, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Solution:

The Beta PDF for a generic random variable X is:

$$f_X(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Based on what we want: $\alpha - 1 = 4 \Rightarrow \alpha = 5$ and $\beta - 1 = 5 \Rightarrow \beta = 6$. Therefore

$$c = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} = \frac{\Gamma(5+6)}{\Gamma(5)\Gamma(6)} = \frac{\Gamma(11)}{\Gamma(5)\Gamma(6)} = \frac{10!}{4!5!} = \frac{3628800}{(24)(120)} = \boxed{1260}.$$

Section 3.4

5. State the MGF of a random variable $X \sim \mathsf{N}(\mu, \sigma^2)$.

Solution:

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}, \quad -\infty < t < \infty.$$

6. Find the value of z_p where $p = 0.95$.

Solution:

By definition, $z_p = \Phi^{-1}(p)$, so

$$z_{0.95} = \Phi^{-1}(0.95) \Rightarrow \Phi(z_{0.95}) = 0.95.$$

This means that

$$\boxed{z_{0.95} = 1.645}.$$

7. Find the value of z_p where $p = 0.9207$.

Solution:

By definition, $z_p = \Phi^{-1}(p)$, so

$$z_{0.9207} = \Phi^{-1}(0.9207) \Rightarrow \Phi(z_{0.9207}) = 0.9207.$$

This means that

$$\boxed{z_{0.9207} = 1.41}.$$

8. If X has the MGF

$$M_X(t) = e^{4t+64t^2},$$

what distribution does X have and what are its parameter values?

Solution:

This is the MGF for a Normally Distributed random variable. In particular, $\mu = 4$ and

$$\frac{\sigma^2}{2} = 64 \Rightarrow \sigma^2 = 128.$$

Therefore, $\boxed{X \sim \mathsf{N}(4, 128)}$.

9. Suppose $X \sim \mathsf{N}(100, 16)$. Find the value of z for:

(a) $x = 90$
Solution:

$$z = \frac{90 - 100}{4} = \boxed{-2.5}$$

(b) $x = 110$
Solution:

$$z = \frac{110 - 100}{4} = \boxed{2.5}$$

(c) $x = 80$
Solution:

$$z = \frac{80 - 100}{4} = \boxed{-5}$$

(d) $x = 105$
Solution:

$$z = \frac{105 - 100}{4} = \boxed{1.25}$$

10. Suppose $X \sim \mathsf{N}(100, 16)$. Find the following probabilities.

(a) $\mathsf{P}[X < 90]$.
Solution:

$$\mathsf{P}[X < 90] = \phi(-2.5) = \boxed{0.0062}.$$

(b) $\mathsf{P}[105 < X < 110]$ ‘
Solution:

$$\mathsf{P}[105 < X < 110] = \Phi(2.5) - \Phi(1.25) = 0.9938 - 0.8944 = \boxed{0.0994}.$$

(c) $\mathsf{P}[X \geq 90]$.
Solution:

$$\mathsf{P}[X \geq 90] = 1 - \mathsf{P}[X < 90] = 1 - \Phi(-2.5) = 1 - 0.0062 = \boxed{0.9938}.$$

(d) $\mathsf{P}[90 < X \leq 105]$
Solution:

$$\mathsf{P}[90 < X \leq 105] = \Phi(1.25) - \Phi(-2.5) = 0.8944 - 0.0062 = \boxed{0.8882}.$$

11. Suppose $X \sim \mathsf{N}(100, 16)$.

(a) Is a value of 90 or smaller likely to occur? Why or why not?

Solution:
It is not likely to happen because $\mathsf{P}[X < 90] = 0.0062$ is very small.

(b) Is a value of 80 or smaller likely to occur? Why or why not?

Solution:
If a value of 90 or smaller is not likely to occur, then seeing a value of 80 or smaller is even less likely to occur. In fact,

$$\mathsf{P}[X < 80] \approx 0.$$

12. If the random variable $X \sim \mathsf{N}(\mu, \sigma^2)$, where $\sigma^2 > 0$, then show that the random variable $(X - \mu)^2/\sigma^2 \sim \chi^2(1)$.

Solution:
First note that

$$\frac{(X-\mu)^2}{\sigma^2} = \left(\frac{X-\mu}{\sigma}\right)^2 = Z^2, \;\; Z \sim \mathsf{N}(0,1).$$

Let $V = Z^2$. The CDF for V is:

$$\mathsf{P}[V \le v] = \mathsf{P}[Z^2 \le v] = \mathsf{P}[-\sqrt{v} < Z < \sqrt{v}];$$

since $-\infty < z < \infty$, we have to take into account both the positive and negative square roots. However, since Z is symmetric, if $v \ge 0$, then

$$\mathsf{P}[V \le v] = 2\int_0^{\sqrt{v}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

Using u-substitution, let $u = z^2$ so that $z = \sqrt{u}$, $du = 2z dz$, and $\frac{1}{2\sqrt{u}} du = dz$. Then

$$\mathsf{P}[V \le v] = 2\int_0^{\sqrt{v}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz = 2\int_0^{v} \frac{1}{\sqrt{2\pi}} e^{-u/2} \cdot \frac{1}{2\sqrt{u}}\, du = \int_0^{v} \frac{1}{\sqrt{2\pi}\sqrt{u}} e^{-u/2}\, du.$$

The CDF for V is:

$$F_V(v) = \begin{cases} 0, & v < 0 \\ \int_0^v \frac{1}{\sqrt{2\pi}\sqrt{u}} e^{-u/2}\, du, & 0 \le v \end{cases}$$

The PDF for V is:

$$f_V(v) = \begin{cases} \frac{1}{\sqrt{\pi}\cdot 2^{1/2}} v^{\frac{1}{2}-1} e^{-v/2}, & 0 < v < \infty \\ 0, & \text{otherwise} \end{cases}$$

Note that $\sqrt{\pi} = \Gamma(1/2)$, and therefore $V \sim \chi^2(1)$.

13. Remember the following corollary:

Corollary 2. *Let $X_1, \ldots, X_n$ be iid random variables with common $\mathsf{N}(\mu, \sigma^2)$ distribution. Let $\bar{X} = n^{-1}\sum_{i=1}^n X_i$. Then $\bar{X} \sim \mathsf{N}(\mu, \sigma^2/n)$.*

Section 3.5

14. Let X and Y have bivariate normal distribution with parameters $\mu_1 = 3$, $\mu_2 = 1$, $\sigma_1^2 = 16$, $\sigma_2^2 = 25$, and $\rho = 3/5$. Determine the following probabilities.

(a) $\mathsf{P}[3 < Y < 8]$

Solution:

One property of the bivariate normal distribution is that the marginal distribution of $Y \sim \mathsf{N}(\mu_2, \sigma_2^2) = \mathsf{N}(1, 25)$.

$$z_1 = \frac{3-1}{5} = \frac{2}{5} = 0.4; \qquad z_2 = \frac{8-1}{5} = \frac{7}{5} = 1.4$$

Then

$$\mathsf{P}[3 < Y < 8] = \mathsf{P}[0.4 < Z < 1.4] = \Phi(1.4) - \Phi(0.4) = 0.9192 - 0.6554 = \boxed{0.2638}.$$

(b) $\mathsf{P}[3 < Y < 8 \mid X = 7]$

Solution:

We know from the bivariate normal distribution that

$$Y \mid X = x \sim \mathsf{N}\left(\mu_2 + \frac{\sigma_2}{\sigma_1}\rho\,(x - \mu_1),\ \ \sigma_2^2\left(1 - \rho^2\right)\right)$$

$$Y \mid X = 7 \sim \mathsf{N}\left(1 + \frac{5}{4}\left(\frac{3}{5}\right)(7-3),\ \ 25\left(1 - \left(\frac{3}{5}\right)^2\right)\right) = \mathsf{N}\left(4, 16\right).$$

From the normal distribution, we have

$$z_1 = \frac{3-4}{4} = -0.25; \qquad z_2 = \frac{8-4}{4} = 1$$

Then

$$\mathsf{P}[3 < Y < 8 \mid X = 7] = \mathsf{P}[-0.25 < Z < 1] = \Phi(1) - \Phi(-0.25) = 0.8413 - 0.4013 = \boxed{0.44}.$$

(c) $\mathsf{P}[-3 < X < 3]$

Solution:

One property of the bivariate normal distribution is that the marginal distribution of $X \sim \mathsf{N}(\mu_1, \sigma_1^2) = \mathsf{N}(3, 16)$.

$$z_1 = \frac{-3-3}{4} = \frac{-6}{4} = -1.5; \qquad z_2 = \frac{3-3}{4} = \frac{0}{4} = 0$$

Then

$$\mathsf{P}[-3 < X < 3] = \mathsf{P}[-1.5 < Z < 0] = \Phi(0) - \Phi(-1.5) = 0.5 - 0.0668 = \boxed{0.4332}.$$

(d) $\mathsf{P}[-3 < X < 3 \mid Y = -4]$

Solution:

We know from the bivariate normal distribution that

$$X \mid Y = y \sim \mathsf{N}\left(\mu_1 + \frac{\sigma_1}{\sigma_2}\rho\,(y - \mu_2)\,,\ \ \sigma_1^2\left(1 - \rho^2\right)\right)$$
$$X \mid Y = -4 \sim \mathsf{N}\left(3 + \frac{4}{5}\left(\frac{3}{5}\right)(-4-1)\,,\ \ 16\left(1 - \left(\frac{3}{5}\right)^2\right)\right) = \mathsf{N}\,(0.6, 10.24)$$

From the normal distribution, we have

$$z_1 = \frac{-3 - 0.6}{\sqrt{10.24}} = -1.125; \qquad z_2 = \frac{3 - 0.6}{\sqrt{10.24}} = 0.75.$$

Then

$$\begin{aligned} \mathsf{P}[-3 < X < 3 \mid Y = -4] &= \mathsf{P}[-1.125 < Z < 0.75] \\ &\approx \mathsf{P}[-1.13 < Z < 0.75] \\ &= \Phi(0.75) - \Phi(-1.13) = 0.7734 - 0.1292 = \boxed{0.6442}. \end{aligned}$$

If we used a graphing calculator, R, etc, we would be able to use the exact z_1 value of -1.125. In that case,

$$\mathsf{P}[-3 < X < 3 \mid Y = -4] = \boxed{0.6431}.$$

15. Let X and Y have bivariate normal distribution with parameters $\mu_1 = 5$, $\mu_2 = 10$, $\sigma_1^2 = 1$, $\sigma_2^2 = 25$, and $\rho > 0$. if $\mathsf{P}[4 < Y < 16 \mid X = 5] = 0.954$, determine ρ.

Solution:

We know from the bivariate normal distribution that

$$Y \mid X = x \sim \mathsf{N}\left(\mu_2 + \frac{\sigma_2}{\sigma_1}\rho\,(x - \mu_1)\,,\ \ \sigma_2^2\left(1 - \rho^2\right)\right)$$
$$Y \mid X = 5 \sim \mathsf{N}\left(10 + \frac{5}{1}\rho(5 - 5),\ \ 25\left(1 - \rho^2\right)\right) = \mathsf{N}\left(10,\ \ 25\left(1 - \rho^2\right)\right).$$

From the normal distribution, we have

$$z_1 = \frac{4 - 10}{5\sqrt{1 - \rho^2}} = \frac{-6}{5\sqrt{1 - \rho^2}}; \qquad z_2 = \frac{16 - 10}{5\sqrt{1 - \rho^2}} = \frac{6}{5\sqrt{1 - \rho^2}}.$$

Then

$$0.954 = \mathsf{P}[4 < Y < 16 \mid X = 5] = \mathsf{P}\left[-\frac{6}{5\sqrt{1 - \rho^2}} < Z < \frac{6}{5\sqrt{1 - \rho^2}}\right]$$
$$= \Phi\left(\frac{6}{5\sqrt{1 - \rho^2}}\right) - \Phi\left(\frac{-6}{5\sqrt{1 - \rho^2}}\right)$$
$$= 1 - 2\Phi\left(\frac{-6}{5\sqrt{1 - \rho^2}}\right).$$

This implies that

$$\Rightarrow 1 - 0.954 = 2\Phi\left(\frac{-6}{5\sqrt{1 - \rho^2}}\right)$$
$$\Rightarrow \frac{0.046}{2} = 0.023 = \Phi\left(\frac{-6}{5\sqrt{1 - \rho^2}}\right).$$

Hence

$$z \approx -2 = \frac{-6}{5\sqrt{1 - \rho^2}}$$
$$\sqrt{1 - \rho^2} = \frac{-6}{5(-2)}$$
$$\sqrt{1 - \rho^2} = 0.6$$
$$1 - \rho^2 = 0.36$$
$$1 - 0.36 = \rho^2$$
$$0.64 = \rho^2$$
$$\frac{4}{5} = 0.8 = \rho \text{ since } \rho > 0.$$

We find that $\boxed{\rho = 4/5}$.

Section 3.6

16. Let T have a t-distribution with 14 degrees of freedom. Determine b so that $\mathsf{P}[-b < T < b] = 0.90$.

Solution:

Since the t-distribution is symmetric,

$$\mathsf{P}[-b < T < b] = 1 - 2\mathsf{P}[T > b] = 0.90.$$

Then

$$1 - 0.90 = 2\mathsf{P}[T > b] \Rightarrow \frac{0.10}{2} = 0.05 = \mathsf{P}[T > b].$$

From the table, for 14 degrees of freedom, $\boxed{b = 1.761}$.

17. Find the corresponding t-values or areas.

(a) Find the t-value such that $\mathsf{P}\left(T > t_{0.01}(16)\right) = 0.01$.

Solution:

$t_{0.01}(16) = 2.583$.

(b) Find the value of $t_{0.975}(14)$.

Solution:

$t_{0.975}(14) = -t_{0.025}(14) = -2.145$.

(c) Find $\mathsf{P}\left(-t_{0.025}(v) < T < t_{0.05}(v)\right)$. v is unknown.

Solution:

We know the area to the left of $-t_{0.025}(v)$ is 0.025. We also know that the area to the right of $t_{0.05}(v)$ is 0.05. The area in between these two values is $1 - 0.025 - 0.05 = \boxed{0.925}$.

(d) Find k such that $\mathsf{P}(T > k) = 0.025$ for 23 degrees of freedom.

Solution:

Our corresponding t-value with 23 degrees of freedom is $k = t_{0.025}(23) = \boxed{2.069}$.

Section 3.7

18. Suppose you have the mixture $0.75\mathsf{N}(0,1) + 0.25\mathsf{N}(1.5,4)$.

   (a) Find its expected value.
   Solution:

$$\mathbf{E}[X] = \sum_{i=1}^{k} p_i \mu_i = 0.75(0) + 0.25(1.5) = \boxed{0.375}.$$

   (b) Find its variance.
   Solution:

$$\begin{aligned}\mathbf{V}[X] &= \sum_{i=1}^{k} p_i \sigma_i^2 + \sum_{i=1}^{k} p_i \left(\mu_i - \bar{\mu}\right)^2 \\ &= 0.75(1) + 0.25(4) + 0.75(0 - 0.375)^2 + 0.25(1.5 - 0.375)^2 = \boxed{2.171875}.\end{aligned}$$

19. Suppose you have the mixture $0.5\mathsf{N}(-1,1) + 0.5\mathsf{N}(1,1)$.

   (a) Find its expected value.
   Solution:

$$\mathbf{E}[X] = \sum_{i=1}^{k} p_i \mu_i = (0.5)(-1) + (0.5)(1) = \boxed{0}$$

   (b) Find its variance.
   Solution:

$$\begin{aligned}\mathbf{V}[X] &= \sum_{i=1}^{k} p_i \sigma_i^2 + \sum_{i=1}^{k} p_i \left(\mu_i - \bar{\mu}\right)^2 \\ &= 0.5(1) + 0.5(1) + 0.5(-1 - 0)^2 + 0.5(1 - 0)^2 = \boxed{2}.\end{aligned}$$

20. Suppose you have the mixture $0.25\mathsf{Pois}(5) + 0.75\chi^2(8)$.

   (a) Find its expected value.

   Solution:
   For the Poisson Distribution, $\mu_1 = 5$. For the χ^2 distribution, $\mu_2 = 8$.

$$\mathbf{E}[X] = \sum_{i=1}^{k} p_i \mu_1 = 0.25(5) + 0.75(8) = \boxed{7.25}.$$

   (b) Find its variance.

   Solution:
   For the Poisson Distribution, $\sigma_1^2 = 5$. For the χ^2 distribution, $\sigma_2^2 = 16$.

$$\begin{aligned}\mathbf{V}[X] &= \sum_{i=1}^{k} p_i \sigma_i^2 + \sum_{i=1}^{k} p_i \left(\mu_i - \bar{\mu}\right)^2 \\ &= 0.25(5) + 0.75(16) + 0.25(5 - 7.25)^2 + 0.75(8 - 7.25)^2 = \boxed{14.9375}.\end{aligned}$$

Section 5.1

21. Suppose $X_1, \ldots, X_n$ is a random sample from a Uniform$(0, \theta)$ distribution. Suppose θ is unknown. An intuitive estimate of θ is the maximum of the sample. Let $Y_n = \max\{X_1, \ldots, X_n\}$. Note: A uniform random variable $X \sim \mathsf{Unif}(a, b)$ has the pdf

$$f(x) = \frac{1}{b-a}, \quad -\infty < a < x < b < \infty.$$

(a) Show that the CDF of Y_n is

$$F_{Y_n}(t) = \begin{cases} 1, & t > \theta \\ \left(\frac{t}{\theta}\right)^n, & 0 < t \leq \theta \\ 0, & t \leq 0. \end{cases}$$

Proof. If $X_i \sim \mathsf{Unif}(0, \theta)$, then

$$f(x_i) = \frac{1}{\theta - 0} = \frac{1}{\theta}, \quad 0 < x < \theta.$$

The X_i's are independent because they come from a random sample.

$$\begin{aligned} \mathsf{P}[Y_n < t] &= \mathsf{P}[X_1 < t, X_2 < t, \ldots, X_n < t] \\ &= \mathsf{P}[X_1 < t]\mathsf{P}[X_2 < t] \cdots \mathsf{P}[X_n < t]; \text{ by independence} \\ &= (\mathsf{P}[X < t])^n; \text{ since the } X_i\text{'s are iid.} \end{aligned}$$

Now we need to find the CDF of X.

$$\mathsf{P}[X \leq x] = \int_0^x \frac{1}{\theta}\, dt = \frac{1}{\theta} t \Big|_0^x = \frac{1}{\theta}(x - 0) = \begin{cases} 0, & x \leq 0 \\ \frac{x}{\theta}, & 0 < x < \theta \\ 1, & \theta \leq x \end{cases}$$

Now,

$$\mathsf{P}[Y_n < t] = (\mathsf{P}[X < t])^n = \left(\frac{t}{\theta}\right)^n.$$

Therefore, the CDF of Y_n is

$$F_{Y_n}(t) = \begin{cases} 1, & t > \theta \\ \left(\frac{t}{\theta}\right)^n, & 0 < t \leq \theta \\ 0, & t \leq 0. \end{cases}$$

□

(b) Find the PDF of Y_n.

Solution:

Find the first derivative of the CDF with respect to t.

$$\frac{d}{dt}\left(\frac{t}{\theta}\right)^n = \frac{d}{dt}\frac{t^n}{\theta^n} = \frac{nt^{n-1}}{\theta^n}.$$

The PDF of Y_n is

$$\boxed{f_{Y_n}(t) = \begin{cases} \frac{n}{\theta^n}t^{n-1}, & 0 < t < \theta \\ 0, & \text{otherwise.} \end{cases}}$$

(c) Show that Y_n is a biased estimator of θ.

Solution:

We need to show that $\mathbf{E}[Y_n] \neq \theta$.

$$\begin{aligned} \mathbf{E}[Y_n] &= \int_0^\theta t \cdot \frac{n}{\theta^n} t^{n-1}\, dt = \frac{n}{\theta^n}\int_0^\theta t^n\, dt = \frac{n}{\theta^n}\left[\frac{t^{n+1}}{n+1}\bigg|_0^\theta\right] \\ &= \frac{n}{(n+1)\theta^n}\left[\theta^{n+1} - 0^{n+1}\right] = \frac{n}{(n+1)\theta^n}\theta^{n+1} \\ &= \boxed{\frac{n}{n+1}\theta \neq \theta}. \end{aligned}$$

(d) Show that $\frac{n+1}{n}Y_n$ is an unbiased estimator of θ.

Solution:

We need to show that $\mathbf{E}\left[\frac{n+1}{n}Y_n\right] = \theta$. From the previous part, we already know that

$$\mathbf{E}[Y_n] = \frac{n}{n+1}\theta.$$

Now,

$$\mathbf{E}\left[\frac{n+1}{n}Y_n\right] = \frac{n+1}{n}\mathbf{E}[Y_n] = \frac{n+1}{n}\cdot\frac{n}{n+1}\theta = \theta.$$

(e) Show that $Y_n \xrightarrow{P} \theta$, i.e. show that Y_n is a consistent estimator of θ.

Solution:

We need to show that $\lim_{n\to\infty} \mathsf{P}\left\{|Y_n - \theta| > \epsilon\right\} = 0$.

$$\begin{aligned}
\mathsf{P}\left\{|Y_n - \theta| > \epsilon\right\} &= \mathsf{P}\left\{|Y_n - \theta| > \epsilon\right\} = \mathsf{P}\left\{\theta - Y_n > \epsilon\right\};\ \text{since } 0 < t < \theta,\ Y_n - \theta < 0 \\
&= \mathsf{P}\left\{-Y_n > \epsilon - \theta\right\} = \mathsf{P}\left\{Y_n < \theta - \epsilon\right\} = F_{Y_n}(\theta - \epsilon) \\
&= \left(\frac{\theta - \epsilon}{\theta}\right)^n = \left(1 - \frac{\epsilon}{\theta}\right)^n.
\end{aligned}$$

Since $\epsilon > 0$ is "small", WLOG assume $0 < \epsilon < \theta$, and we have $0 < \epsilon/\theta < 1$. This means that

$$1 - \frac{\epsilon}{\theta} < 1.$$

Hence $\left(1 - \frac{\epsilon}{\theta}\right)^n \to 0$ as $n \to \infty$. Therefore

$$\lim_{n\to\infty} \mathsf{P}\left\{|Y_n - \theta| > \epsilon\right\} = \left(1 - \frac{\epsilon}{\theta}\right)^n = 0.$$

Therefore, $Y_n \xrightarrow{P} \theta$.

(f) Show that $\frac{n+1}{n} Y_n$ is a consistent estimator of θ.

Solution:

We need to show that $\lim_{n\to\infty} \mathsf{P}\left\{\left|\frac{n+1}{n} Y_n - \theta\right| > \epsilon\right\} = 0$.

$$\begin{aligned}
\mathsf{P}\left\{\left|\frac{n+1}{n} Y_n - \theta\right| > \epsilon\right\} &= \mathsf{P}\left\{\left|\frac{n+1}{n}\right|\left|Y_n - \frac{n}{n+1}\theta\right| > \epsilon\right\};\ \text{note } \left|\frac{n+1}{n}\right| = \frac{n+1}{n} \\
&= \mathsf{P}\left\{\left|Y_n - \frac{n}{n+1}\theta\right| > \frac{n}{n+1}\epsilon\right\};\ \text{note } \mathbf{E}[Y_n] = \frac{n}{n+1}\theta \\
&\le \frac{\mathbf{V}[Y_n]}{\left(\frac{n}{n+1}\epsilon\right)^2};\ \text{Chebyshev's Inequality}
\end{aligned}$$

We need to find $\mathbf{V}[Y_n]$.

$$\begin{aligned}
\mathbf{E}\left[Y_n\right]^2 &= \int_0^\theta t^2 \cdot \frac{n}{\theta^n} t^{n-1}\, dt = \frac{n}{\theta^n} \int_0^\theta t^{n+1}\, dt \\
&= \frac{n}{\theta^n}\left[\left.\frac{t^{n+2}}{n+2}\right|_0^\theta\right] = \frac{n}{(n+2)\theta^n}\left[\theta^{n+2} - 0\right] = \frac{n}{n+2}\theta^2 \\
\mathbf{V}[Y_n] &= \frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\theta\right)^2 = \frac{n}{n+2}\theta^2 - \frac{n^2}{(n+1)^2}\theta^2 = \left[\frac{n}{n+2} - \frac{n^2}{(n+1)^2}\right]\theta^2.
\end{aligned}$$

Then

$$\mathsf{P}\left\{\left|\frac{n+1}{n}Y_n-\theta\right|>\epsilon\right\}\le\frac{\mathbf{V}[Y_n]}{\left(\frac{n}{n+1}\epsilon\right)^2}=\frac{\left[\frac{n}{n+2}-\frac{n^2}{(n+1)^2}\right]\theta^2}{\frac{n^2}{(n+1)^2}\epsilon^2}$$
$$=\frac{\theta^2}{\epsilon^2}\cdot\left[\frac{n}{n+2}-\frac{n^2}{(n+1)^2}\right]\cdot\frac{(n+1)^2}{n^2}$$
$$=\frac{\theta^2}{\epsilon^2}\cdot\left[\frac{(n+1)^2}{n(n+2)}-1\right]=\frac{\theta^2}{\epsilon^2}\cdot\left[\frac{n^2+2n+1}{n^2+2n}-1\right].$$

Recall from Calculus (L'Hopital's Rule) that

$$\lim_{n\to\infty}\frac{n^2+2n+1}{n^2+2n}=\lim_{n\to\infty}\frac{2n+2}{2n+2}=\lim_{n\to\infty}1=1.$$

We have

$$\mathsf{P}\left\{\left|\frac{n+1}{n}Y_n-\theta\right|>\epsilon\right\}\le\frac{\mathbf{V}[Y_n]}{\left(\frac{n}{n+1}\epsilon\right)^2}=\frac{\theta^2}{\epsilon^2}\cdot\left[\frac{n^2+2n+1}{n^2+2n}-1\right]$$
$$\to\frac{\theta^2}{\epsilon^2}\cdot[1-1]\text{ as }n\to\infty$$
$$=\frac{\theta^2}{\epsilon^2}(0)=0.$$

Therefore $\frac{n+1}{n}Y_n\xrightarrow{P}\theta$.

22. Suppose $X_1,\ldots,X_n$ is a random sample from a Uniform$(0,\theta)$ distribution. Suppose θ is unknown. Show that $\bar{X}_n$ is a consistent estimator of $\theta/2$.

Solution:

We need to show that $\bar{X}_n\xrightarrow{P}\theta/2$. Recall that the CDF of X is:

$$\mathsf{P}[X\le x]=\int_0^x\frac{1}{\theta}\,dt=\frac{1}{\theta}t\Big|_0^x=\frac{1}{\theta}(x-0)=\begin{cases}0, & x\le 0\\ \frac{x}{\theta}, & 0<x<\theta\\ 1, & \theta\le x\end{cases}$$

The PDF of X is

$$f(x)=\begin{cases}\frac{1}{\theta}, & 0<x<\theta\\ 0, & \text{otherwise.}\end{cases}$$

The expected value of X is

$$\mathbf{E}[X]=\int_0^\theta x\cdot\frac{1}{\theta}\,dx=\frac{1}{\theta}\left[\frac{x^2}{2}\Big|_0^\theta\right]=\frac{1}{2\theta}\left(\theta^2-0\right)=\frac{1}{2}\theta.$$

By the WLLN, $\bar{X}_n\xrightarrow{P}\frac{\theta}{2}$.

Section 5.2

23. Suppose $X_1, \dots, X_n$ is a random sample from a Uniform$(0, \theta)$ distribution. Suppose θ is unknown. An intuitive estimate of θ is the maximum of the sample. Let $Y_n = \max\{X_1, \dots, X_n\}$. Consider the random variable $Z_n = n(\theta - Y_n)$. Let $t \in (0, n\theta)$. Show that $Z_n \xrightarrow{D} Z$, where $Z \sim \mathsf{Exp}(\theta)$.

Proof. Recall that the CDF of Y_n is

$$F_{Y_n}(t) = \begin{cases} 1, & t > \theta \\ \left(\frac{t}{\theta}\right)^n, & 0 < t \le \theta \\ 0, & t \le 0. \end{cases}$$

$$\begin{aligned}
\mathsf{P}[Z_n \le t] &= \mathsf{P}[n(\theta - Y_n) \le t] = \mathsf{P}\left[\theta - Y_n \le \frac{t}{n}\right] = \mathsf{P}\left[-Y_n \le \frac{t}{n} - \theta\right] \\
&= \mathsf{P}\left[Y_n \ge \theta - \frac{t}{n}\right] = 1 - \mathsf{P}\left[Y_n \le \theta - \frac{t}{n}\right] = 1 - \left(\frac{\theta - \frac{t}{n}}{\theta}\right)^n \\
&= 1 - \left(1 - \frac{t}{n\theta}\right)^n = 1 - \left(1 + \frac{-t/\theta}{n}\right)^n \to 1 - e^{(-t/\theta)(1)} \text{ as } n \to \infty \\
&= 1 - e^{-t/\theta}
\end{aligned}$$

Note that $1 - e^{-t/\theta}$ is the CDF of an exponential random variable with mean θ. Therefore $Z_n \xrightarrow{D} \mathsf{Exp}(\theta)$. ☐

24. Let $Z_n \sim \chi^2(n)$. Find the limiting distribution of the random variable $Y_n = (Z_n - n)/\sqrt{2n}$ by using Moment Generating Functions and Taylor's Expansion.

Solution:

Recall that $M_{Z_n}(t) = \frac{1}{(1-2t)^{n/2}}$, $t < 1/2$. We also have $\mathbf{E}[Z_n] = n$; $\quad \mathbf{V}[Z_n] = 2n$.

$$\begin{aligned}
M_{Y_n}(t) &= \mathbf{E}\left[e^{tY_n}\right] = \mathbf{E}\left[\exp\left\{t \cdot \frac{Z_n - n}{\sqrt{2n}}\right\}\right] = \mathbf{E}\left[\exp\left\{\frac{t}{\sqrt{2n}} Z_n\right\} \exp\left\{\frac{-nt}{\sqrt{2n}}\right\}\right] \\
&= e^{-t\sqrt{n/2}} M_{Z_n}\left(\frac{t}{\sqrt{2n}}\right) = e^{-t\sqrt{n/2}} \left(1 - 2 \cdot \frac{t}{\sqrt{2n}}\right)^{-n/2} \\
&= e^{-t\sqrt{n/2}} \left(1 - t\sqrt{\frac{2}{n}}\right)^{-n/2}, \quad t < \sqrt{\frac{n}{2}} \\
&= e^{-t\sqrt{n/2}\underbrace{\sqrt{n/2}\sqrt{2/n}}_{=1}} \left(1 - t\sqrt{\frac{2}{n}}\right)^{-n/2}, \quad t < \sqrt{\frac{n}{2}} \\
&= e^{t\sqrt{2/n}(-n/2)} \left(1 - t\sqrt{\frac{2}{n}}\right)^{-n/2}, \quad t < \sqrt{\frac{n}{2}}
\end{aligned}$$

$$= \left[e^{t\sqrt{2/n}}\left(1 - t\sqrt{\frac{2}{n}}\right)\right]^{-n/2}, \;\; t < \sqrt{\frac{n}{2}}.$$

By Taylor's Expansion, there exists a number $c(n)$, between 0 and $t\sqrt{2/n}$ such that

$$e^{t\sqrt{2/n}} = 1 + t\sqrt{\frac{2}{n}} + \frac{1}{2}\left(t\sqrt{\frac{2}{n}}\right)^2 + \frac{e^{c(n)}}{6}\left(t\sqrt{\frac{2}{n}}\right)^3.$$

Then

$$\begin{aligned}
e^{t\sqrt{2/n}}\left(1 - t\sqrt{\frac{2}{n}}\right) &= \left(1 + t\sqrt{\frac{2}{n}} + \frac{1}{2}\left(t\sqrt{\frac{2}{n}}\right)^2 + \frac{e^{c(n)}}{6}\left(t\sqrt{\frac{2}{n}}\right)^3\right)\left(1 - t\sqrt{\frac{2}{n}}\right) \\
&= 1 + t\sqrt{\frac{2}{n}} + \frac{1}{2}\left(t\sqrt{\frac{2}{n}}\right)^2 + \frac{e^{c(n)}}{6}\left(t\sqrt{\frac{2}{n}}\right)^3 \\
&\quad - t\sqrt{\frac{2}{n}} - t^2 \cdot \frac{2}{n} - \frac{t}{2}\sqrt{\frac{2}{n}}\left(t\sqrt{\frac{2}{n}}\right)^2 - \frac{e^{c(n)}}{6}\left(t\sqrt{\frac{2}{n}}\right)^3 \cdot t\sqrt{\frac{2}{n}} \\
&= 1 + \frac{t^2}{2} \cdot \frac{2}{n} + \frac{e^{c(n)}t^3}{6} \cdot \frac{2}{n}\sqrt{\frac{2}{n}} - \frac{2t^2}{n} - \frac{t}{2}\sqrt{\frac{2}{n}} \cdot t^2 \cdot \frac{2}{n} - \frac{t^4 e^{c(n)}}{6} \cdot \frac{2}{n} \cdot \frac{2}{n} \\
&= 1 + \frac{t^2}{n} + \frac{t^3 e^{c(n)}\sqrt{2}}{3n\sqrt{n}} - \frac{2t^2}{n} - \frac{t^3\sqrt{2}}{n\sqrt{n}} - \frac{2t^4 e^{c(n)}}{3n^2} \\
&= 1 + \frac{-t^2}{n} + \frac{\psi(n)}{n},
\end{aligned}$$

where

$$\psi(n) = \frac{t^3 e^{c(n)}\sqrt{2}}{3\sqrt{n}} - \frac{t^3\sqrt{2}}{\sqrt{n}} - \frac{2t^4 e^{c(n)}}{3n}.$$

This means that

$$M_{Y_n}(t) = \left[1 + \frac{-t^2}{n} + \frac{\psi(n)}{n}\right]^{-n/2}.$$

Note that $t\sqrt{2/n} \to 0$ as $n \to \infty$. This means that $c(n) \to 0$ and $e^{c(n)} \to 1$ as $n \to \infty$. For every fixed value of t,

$$\lim_{n\to\infty} \psi(n) = 0 - 0 - 0 = 0.$$

Recall

$$\lim_{n\to\infty}\left[1 + \frac{b}{n} + \frac{\psi(n)}{n}\right]^{cn} = \lim_{n\to\infty}\left(1 + \frac{b}{n}\right)^{cn} = e^{bc},$$

where b and c do not depend on n and where $\lim_{n\to\infty}\psi(n) = 0$. This gives us the conclusion that

$$\lim_{n\to\infty} M_{Y_n}(t) = e^{(-t^2)(-1/2)} = e^{t^2/2} = e^{0t + t^2/2}, \;\; \forall t.$$

This is the MGF of the Standard Normal Distribution. Therefore

$$Y_n \xrightarrow{D} N(0,1).$$

Section 5.3

25. Let $\bar{X}$ denote the mean of a random sample of size 128 from a Gamma Distribution with $\alpha = 2$ and $\beta = 4$. Approximate $\mathsf{P}[7 < \bar{X} < 9]$.

Solution:

For a Gamma Distribution,

$$\mu = \alpha\beta = 2(4) = 8$$
$$\sigma^2 = \alpha\beta^2 = 2(4)^2 = 2(16) = 32$$

Then

$$\begin{aligned}\mathsf{P}[7 < \bar{X} < 9] &= \mathsf{P}\left[\frac{7-8}{\sqrt{32}/\sqrt{128}} < \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < \frac{9-8}{\sqrt{32}/\sqrt{128}}\right] \\ &= \mathsf{P}\left[-2 < \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < 2\right] \approx \mathsf{P}[-2 < Z < 2] \\ &= \Phi(2) - \Phi(-2) = 0.9772 - 0.0228 \\ &= \boxed{0.9544}.\end{aligned}$$

26. Let $Y \sim \mathsf{Bin}\left(72, \frac{1}{3}\right)$. Approximate $\mathsf{P}[22 \le Y \le 28]$.

Solution:

For the Binomial Distribution,

$$\mu = np = 72\left(\frac{1}{3}\right) = 24$$
$$\sigma^2 = np(1-p) = 72\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = 16$$
$$\sigma = \sqrt{16} = 4$$

Then

$$\begin{aligned}\mathsf{P}[22 \le Y \le 28] = \mathsf{P}[21.5 < Y < 28.5] &= \mathsf{P}\left[\frac{21.5-24}{4} < \frac{Y-24}{4} < \frac{28.5-24}{4}\right] \\ &= \mathsf{P}\left[-0.625 < \frac{Y-24}{4} < 1.125\right] \approx \mathsf{P}[-0.63 < Z < 1.13] \\ &= \Phi(1.13) - \Phi(-0.63) = 0.8708 - 0.2643 \\ &= \boxed{0.6065}.\end{aligned}$$

If you use technology (graphing calculator, R, etc), then

$$\mathsf{P}[22 \le Y \le 28] \approx \boxed{0.6037}.$$

27. Let $Y \sim \mathsf{Bin}\left(400, \frac{1}{5}\right)$. Compute an approximate value of $\mathsf{P}\left[0.25 < \frac{Y}{400}\right]$.

Solution:

For the Binomial Distribution,

$$\begin{aligned}\mu &= np = 400\left(\frac{1}{5}\right) = 80\\ \sigma^2 &= np(1-p) = 400\left(\frac{1}{5}\right)\left(\frac{4}{5}\right) = 64\\ \sigma &= \sqrt{64} = 8.\end{aligned}$$

Then

$$\begin{aligned}\mathsf{P}\left[0.25 < \frac{Y}{400}\right] &= \mathsf{P}\left[0.25(400) < Y\right] = \mathsf{P}[Y > 100] = \mathsf{P}[Y > 100.5]\\ &= \mathsf{P}\left[\frac{Y-80}{8} > \frac{100.5-80}{8}\right] = \mathsf{P}\left[\frac{Y-80}{8} > 2.5625\right]\\ &\approx \mathsf{P}[Z > 2.56] = 1 - \mathsf{P}[Z \leq 2.56] = 1 - \Phi(2.56) = 1 - 0.9948\\ &= \boxed{0.0052}.\end{aligned}$$

If you use technology (graphing calculator, R, etc), then

$$\mathsf{P}\left[0.25 < \frac{Y}{400}\right] \approx 1 - \Phi(2.5625) = 1 - 0.9948 = \boxed{0.0052}.$$

28. If $Y \sim \mathsf{Bin}\left(100, \frac{1}{2}\right)$, approximate the value of $\mathsf{P}[Y = 50]$.

Solution:

For the Binomial Distribution,

$$\begin{aligned}\mu &= np = 100\left(\frac{1}{2}\right) = 50\\ \sigma^2 &= np(1-p) = 100\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 25\\ \sigma &= \sqrt{25} = 5.\end{aligned}$$

Then

$$\begin{aligned}\mathsf{P}[Y = 50] &= \mathsf{P}[49.5 < Y < 50.5] = \mathsf{P}\left[\frac{49.5-50}{5} < \frac{Y-50}{5} < \frac{50.5-50}{5}\right]\\ &= \mathsf{P}\left[-0.1 < \frac{Y-50}{5} < 0.1\right] \approx \mathsf{P}[-0.1 < Z < 0.1]\\ &= \Phi(0.1) - \Phi(-0.1) = 0.5398 - 0.4602\\ &= \boxed{0.0796}.\end{aligned}$$