

# Characteristic Classes on Determinantal Varieties

Xiping Zhang  
xzhang@math.fsu.edu

Midwest Algebraic Geometry Graduate Conference  
University of Illinois at Chicago

April 22, 2017

# Determinantal Variety

- \* Let  $K$  be an algebraic closed field of characteristic 0.
- \* Let  $M = \mathbb{P}(\operatorname{Hom}(K^n, K^m)) = \mathbb{P}^{mn-1}$  be the projective space.
- \* Let  $\tau_{m,n,k} = \{\varphi \in M \mid \dim \ker \varphi \geq k\}$  be the Determinantal Variety.
- \* Denote the locally closed subset  $\tau_{m,n,k} \setminus \tau_{m,n,k+1}$  by  $\tau_{m,n,k}^\circ$ .

# Formula

For  $k \geq 1$ ,  $i, p = 0, 1 \cdots m(n-k)$ , let

$$A_{i,p}(k) = A_{i,p}(m, n, k) := \int_{G(k,n)} c(S^\vee \otimes Q) c_i(Q^{\vee m}) c_{p-i}(S^{\vee m})$$

$$B_{i,p}(k) = B_{i,p}(m, n, k) := \binom{m(n-k) - p}{i - p}$$

Convention:  $\binom{a}{b} = 0$  for  $a < b$  or  $a < 0$  or  $b < 0$ .

# Formula

Let

$$A(k) = A(m, n, k) = [A_{i,p}(k)]_{i,p}$$

$$B(k) = B(m, n, k) = [B_{i,p}(k)]_{i,p}$$

$$\mathcal{H}(k) = \mathcal{H}(m, n, k) = \begin{pmatrix} H^{mk} & H^{mk+1} & \dots & H^{mn} \\ H^{mk-1} & H^{mk} & \dots & H^{mn-1} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & H^{mk-1} & H^{mk} \end{pmatrix}.$$

be  $m(n-k)+1 \times m(n-k)+1$  matrices.

# Formula

## Theorem (Zhang, 2016)

For  $k \geq 1$ , the Chern-Mather class of  $\tau_{m,n,k}$  is

$$c_M(\tau_{m,n,k}) = \text{Trace}(A(k) \cdot \mathcal{H}(k) \cdot B(k)).$$

## corollary

For  $k \geq 1$ , the Chern-Schwartz-MacPherson class of  $\tau_{m,n,k}$  is

$$c_{SM}(\tau_{m,n,k}) = \sum_{i=k}^{n-1} (-1)^{i-k} \binom{j-1}{k-1} c_M(\tau_{m,n,i}).$$

## Chern-Mather Class

1.  $\tau_{3,3,k}$ 

| Table               | $\mathbb{P}^0$ | $\mathbb{P}^1$ | $\mathbb{P}^2$ | $\mathbb{P}^3$ | $\mathbb{P}^4$ | $\mathbb{P}^5$ | $\mathbb{P}^6$ | $\mathbb{P}^7$ | $\mathbb{P}^8$ |
|---------------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $c_M(\tau_{3,3,0})$ | 9              | 36             | 84             | 126            | 126            | 84             | 36             | 9              | 1              |
| $c_M(\tau_{3,3,1})$ | 18             | 54             | 102            | 126            | 102            | 54             | 18             | 3              | 0              |
| $c_M(\tau_{3,3,2})$ | 9              | 18             | 24             | 18             | 6              | 0              | 0              | 0              | 0              |

2.  $\tau_{4,3,k}$ 

| Table               | $\mathbb{P}^0$ | $\mathbb{P}^1$ | $\mathbb{P}^2$ | $\mathbb{P}^3$ | $\mathbb{P}^4$    | $\mathbb{P}^5$    |
|---------------------|----------------|----------------|----------------|----------------|-------------------|-------------------|
| $c_M(\tau_{4,3,0})$ | 12             | 66             | 220            | 495            | 792               | 924               |
| $c_M(\tau_{4,3,1})$ | 24             | 96             | 248            | 444            | 564               | 514               |
| $c_M(\tau_{4,3,2})$ | 12             | 30             | 52             | 57             | 36                | 10                |
| Table               | $\mathbb{P}^6$ | $\mathbb{P}^7$ | $\mathbb{P}^8$ | $\mathbb{P}^9$ | $\mathbb{P}^{10}$ | $\mathbb{P}^{11}$ |
| $c_M(\tau_{4,3,0})$ | 792            | 495            | 220            | 66             | 12                | 1                 |
| $c_M(\tau_{4,3,1})$ | 336            | 153            | 44             | 6              | 0                 | 0                 |
| $c_M(\tau_{4,3,2})$ | 0              | 0              | 0              | 0              | 0                 | 0                 |

Chern-Mather Class of  $\tau_{4,4,k}$ 

| Table               | $\mathbb{P}^0$ | $\mathbb{P}^1$ | $\mathbb{P}^2$    | $\mathbb{P}^3$    | $\mathbb{P}^4$    | $\mathbb{P}^5$    | $\mathbb{P}^6$    | $\mathbb{P}^7$    |
|---------------------|----------------|----------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| $c_M(\tau_{4,4,1})$ | 48             | 288            | 1128              | 3168              | 6672              | 10816             | 13716             | 13716             |
| $c_M(\tau_{4,4,2})$ | 48             | 216            | 672               | 1524              | 2592              | 3368              | 3376              | 2602              |
| $c_M(\tau_{4,4,3})$ | 16             | 48             | 104               | 152               | 144               | 80                | 20                | 0                 |
| Table               | $\mathbb{P}^8$ | $\mathbb{P}^9$ | $\mathbb{P}^{10}$ | $\mathbb{P}^{11}$ | $\mathbb{P}^{12}$ | $\mathbb{P}^{13}$ | $\mathbb{P}^{14}$ | $\mathbb{P}^{15}$ |
| $c_M(\tau_{4,4,1})$ | 10816          | 6672           | 3168              | 1128              | 288               | 48                | 4                 | 0                 |
| $c_M(\tau_{4,4,2})$ | 1504           | 616            | 160               | 20                | 0                 | 0                 | 0                 | 0                 |
| $c_M(\tau_{4,4,3})$ | 0              | 0              | 0                 | 0                 | 0                 | 0                 | 0                 | 0                 |

Chern-Schwartz-MacPherson Class of  $\tau_{4,4,s}^\circ$ 

| Table                        | $\mathbb{P}^0$ | $\mathbb{P}^1$ | $\mathbb{P}^2$    | $\mathbb{P}^3$    | $\mathbb{P}^4$    | $\mathbb{P}^5$    | $\mathbb{P}^6$    | $\mathbb{P}^7$    |
|------------------------------|----------------|----------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| $c_{SM}(\tau_{4,4,0}^\circ)$ | 0              | 0              | 0                 | 24                | 144               | 480               | 1080              | 1756              |
| $c_{SM}(\tau_{4,4,1}^\circ)$ | 0              | 0              | 96                | 576               | 1920              | 4320              | 7024              | 8512              |
| $c_{SM}(\tau_{4,4,2}^\circ)$ | 0              | 72             | 360               | 1068              | 2160              | 3128              | 3316              | 2602              |
| $c_{SM}(\tau_{4,4,3}^\circ)$ | 16             | 48             | 104               | 152               | 144               | 80                | 20                | 0                 |
| Table                        | $\mathbb{P}^8$ | $\mathbb{P}^9$ | $\mathbb{P}^{10}$ | $\mathbb{P}^{11}$ | $\mathbb{P}^{12}$ | $\mathbb{P}^{13}$ | $\mathbb{P}^{14}$ | $\mathbb{P}^{15}$ |
| $c_{SM}(\tau_{4,4,0}^\circ)$ | 2128           | 1952           | 1360              | 712               | 272               | 72                | 12                | 1                 |
| $c_{SM}(\tau_{4,4,1}^\circ)$ | 7808           | 5440           | 2848              | 1088              | 288               | 48                | 4                 | 0                 |
| $c_{SM}(\tau_{4,4,2}^\circ)$ | 1504           | 616            | 160               | 20                | 0                 | 0                 | 0                 | 0                 |
| $c_{SM}(\tau_{4,4,3}^\circ)$ | 0              | 0              | 0                 | 0                 | 0                 | 0                 | 0                 | 0                 |



Chern-Schwartz-MacPherson Class of  $\tau_{5,3,k}^\circ$ 

| Table                        | $\mathbb{P}^0$ | $\mathbb{P}^1$ | $\mathbb{P}^2$    | $\mathbb{P}^3$    | $\mathbb{P}^4$    | $\mathbb{P}^5$    | $\mathbb{P}^6$    | $\mathbb{P}^7$ |
|------------------------------|----------------|----------------|-------------------|-------------------|-------------------|-------------------|-------------------|----------------|
| $c_{SM}(\tau_{5,3,0}^\circ)$ | 0              | 0              | 60                | 360               | 1170              | 2550              | 3990              | 4620           |
| $c_{SM}(\tau_{5,3,1}^\circ)$ | 0              | 60             | 300               | 870               | 1710              | 2390              | 2430              | 1815           |
| $c_{SM}(\tau_{5,3,2}^\circ)$ | 15             | 45             | 95                | 135               | 123               | 65                | 15                | 0              |
| Table                        | $\mathbb{P}^8$ | $\mathbb{P}^9$ | $\mathbb{P}^{10}$ | $\mathbb{P}^{11}$ | $\mathbb{P}^{12}$ | $\mathbb{P}^{13}$ | $\mathbb{P}^{14}$ |                |
| $c_{SM}(\tau_{5,3,0}^\circ)$ | 4015           | 2625           | 1275              | 445               | 105               | 15                | 1                 |                |
| $c_{SM}(\tau_{5,3,1}^\circ)$ | 990            | 378            | 90                | 10                | 0                 | 0                 | 0                 |                |
| $c_{SM}(\tau_{5,3,2}^\circ)$ | 0              | 0              | 0                 | 0                 | 0                 | 0                 | 0                 |                |

# Conormal and Characteristic Cycles

Let  $N = mn - 1$ . Write the Chern-Mather class and the Chern-Schwartz-MacPherson class of  $\tau_{m,n,k}$  as

$$c_M(\tau_{m,n,k}) = \sum_{l=0}^N \alpha_l [\mathbb{P}^l],$$

and

$$c_{SM}(\tau_{m,n,k}) = \sum_{l=0}^N \beta_l [\mathbb{P}^l].$$

# Conormal and Characteristic Cycles

The *Conormal cycle* and the *Characteristic cycle* of  $\tau_{m,n,k}$  are  $N - 1$  dimensional cycles in  $\mathbb{P}(T_{\mathbb{P}^N}^*) \subset \mathbb{P}^N \times \mathbb{P}^N$ .

$$Con(\tau_{m,n,k}) = \sum_{j=1}^{mn-2} \sum_{l=j-1}^{mn-2} (-1)^l \alpha_l \binom{l+1}{j} h_1^{mn-j} h_2^j \cap [\mathbb{P}^N \times \mathbb{P}^N].$$

and

$$Ch(\tau_{m,n,k}) = \sum_{j=1}^{mn-2} \sum_{l=j-1}^{mn-2} (-1)^l \beta_l \binom{l+1}{j} h_1^{mn-j} h_2^j \cap [\mathbb{P}^N \times \mathbb{P}^N].$$

# Conormal Cycle

| Table               | $h_1^{15}h_2$ | $h_1^{14}h_2^2$ | $h_1^{13}h_2^3$ | $h_1^{12}h_2^4$ | $h_1^{11}h_2^5$ | $h_1^{10}h_2^6$ | $h_1^9h_2^7$ |
|---------------------|---------------|-----------------|-----------------|-----------------|-----------------|-----------------|--------------|
| $Con(\tau_{4,4,1})$ | 0             | 0               | 0               | 0               | 0               | 0               | 0            |
| $Con(\tau_{4,4,2})$ | 0             | 0               | 0               | -20             | -80             | -176            | -256         |
| $Con(\tau_{4,4,3})$ | 4             | 12              | 36              | 68              | 84              | 60              | 20           |

  

| Table               | $h_1^8h_2^8$ | $h_1^7h_2^9$ | $h_1^6h_2^{10}$ | $h_1^5h_2^{11}$ | $h_1^4h_2^{12}$ | $h_1^3h_2^{13}$ | $h_1^2h_2^{14}$ | $h_1h_2^{15}$ |
|---------------------|--------------|--------------|-----------------|-----------------|-----------------|-----------------|-----------------|---------------|
| $Con(\tau_{4,4,1})$ | 0            | 20           | 60              | 84              | 68              | 36              | 12              | 4             |
| $Con(\tau_{4,4,2})$ | -286         | -256         | -176            | -80             | -20             | 0               | 0               | 0             |
| $Con(\tau_{4,4,3})$ | 0            | 0            | 0               | 0               | 0               | 0               | 0               | 0             |

# Conormal Cycle

| Table               | $h_1^{19}h_2$ | $h_1^{18}h_2^2$ | $h_1^{17}h_2^3$ | $h_1^{16}h_2^4$ | $h_1^{15}h_2^5$ | $h_1^{14}h_2^6$ | $h_1^{13}h_2^7$ | $h_1^{12}h_2^8$ | $h_1^{11}h_2^9$ |
|---------------------|---------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $Con(\tau_{5,4,1})$ | 0             | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               |
| $Con(\tau_{5,4,2})$ | 0             | 0               | 0               | 0               | 0               | -50             | -240            | -595            | -960            |
| $Con(\tau_{5,4,3})$ | 0             | -10             | -40             | -105            | -176            | -190            | -120            | -35             | 0               |

  

| Table               | $h_1^{10}h_2^{10}$ | $h_1^9h_2^{11}$ | $h_1^8h_2^{12}$ | $h_1^7h_2^{13}$ | $h_1^6h_2^{14}$ | $h_1^5h_2^{15}$ | $h_1^4h_2^{16}$ | $h_1^3h_2^{17}$ | $h_1^2h_2^{18}$ | $h_1h_2^{19}$ |
|---------------------|--------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|---------------|
| $Con(\tau_{5,4,1})$ | 0                  | 0               | -35             | -120            | -190            | -176            | -105            | -40             | -10             | 0             |
| $Con(\tau_{5,4,2})$ | -1116              | -960            | -595            | -240            | -50             | 0               | 0               | 0               | 0               | 0             |
| $Con(\tau_{5,4,3})$ | 0                  | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0               | 0             |

# Characteristic Cycle

| Table              | $h_1^{11}h_2$ | $h_1^{10}h_2^2$ | $h_1^9h_2^3$ | $h_1^8h_2^4$    | $h_1^7h_2^5$  | $h_1^6h_2^6$ |
|--------------------|---------------|-----------------|--------------|-----------------|---------------|--------------|
| $Ch(\tau_{4,3,1})$ | 0             | 6               | 16           | 27              | 24            | 0            |
| $Ch(\tau_{4,3,2})$ | 0             | -6              | -16          | -27             | -24           | -10          |
| Table              | $h_1^5h_2^7$  | $h_1^4h_2^8$    | $h_1^3h_2^9$ | $h_1^2h_2^{10}$ | $h_1h_2^{11}$ |              |
| $Ch(\tau_{4,3,1})$ | -24           | -27             | -16          | -6              | 0             |              |
| $Ch(\tau_{4,3,2})$ | 0             | 0               | 0            | 0               | 0             |              |

$\tau_{5,4,k}$ 

| Table              | $h_1^{19} h_2$   | $h_1^{18} h_2^2$ | $h_1^{17} h_2^3$ | $h_1^{16} h_2^4$ | $h_1^{15} h_2^5$ | $h_1^{14} h_2^6$ | $h_1^{13} h_2^7$ | $h_1^{12} h_2^8$ | $h_1^{11} h_2^9$ | $h_1^{10} h_2^{10}$ |
|--------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|---------------------|
| $Ch(\tau_{5,4,1})$ | 0                | -10              | -40              | -105             | -176             | -140             | 120              | 560              | 960              | 1116                |
| $Ch(\tau_{5,4,2})$ | 0                | 20               | 80               | 210              | 352              | 330              | 0                | -525             | -960             | -1116               |
| $Ch(\tau_{5,4,3})$ | 0                | -10              | -40              | -105             | -176             | -190             | -120             | -35              | 0                | 0                   |
| Table              | $h_1^9 h_2^{11}$ | $h_1^8 h_2^{12}$ | $h_1^7 h_2^{13}$ | $h_1^6 h_2^{14}$ | $h_1^5 h_2^{15}$ | $h_1^4 h_2^{16}$ | $h_1^3 h_2^{17}$ | $h_1^2 h_2^{18}$ | $h_1 h_2^{19}$   |                     |
| $Ch(\tau_{5,4,1})$ | 960              | 560              | 120              | -140             | -176             | -105             | -40              | -10              | 0                |                     |
| $Ch(\tau_{5,4,2})$ | -960             | -595             | -240             | -50              | 0                | 0                | 0                | 0                | 0                |                     |
| $Ch(\tau_{5,4,3})$ | 0                | 0                | 0                | 0                | 0                | 0                | 0                | 0                | 0                |                     |

Thank You !