

Counting Parentheses

Parentheses are correctly written when all opened parentheses get closed, and you never close a parenthesis before it gets opened. For example, “ $(())$ ” or “ $()()$ ” are ok, but “ $()()$ ” is not ok because we closed a parenthesis at the beginning before anything was open, and “ $()(($ ” is not ok because there are more open parentheses than closed parentheses.

Question 1. There are two ways to write two sets of parentheses correctly, namely “ $()()$ ” and “ $(())$ ”. How many ways are there to write three sets of parentheses correctly? List all of them out.

Question 2. How many different ways can you write four sets of parentheses correctly? List all of them out.

Counting Mountain Ranges

Question 3. How many ways can you make a “mountain range” picture with two upstrokes and two downstrokes without going below your start point? (One example of a mountain range picture is $/\backslash/\backslash$). Draw all of them.

Question 4. How many ways can you make a mountain range picture with three upstrokes and three downstrokes? Draw all of them.

Question 5. How many ways can you make a mountain range picture with four upstrokes and four downstrokes? Draw all of them.

Counting Handshakes

Question 6. If 4 people sit at a table, how many ways can they each shake hands with one other person (at the same time) so that nobody's arms cross? Draw a picture or graph for each possibility.

Question 7. If 6 people sit at a table, how many ways can they each shake hands with one other person (at the same time) so that nobody's arms cross? Draw a picture or graph for each possibility.

Question 8. If 8 people sit at a table, how many ways can they each shake hands with one other person (at the same time) so that nobody's arms cross? Draw a picture or graph for each possibility.

Counting Ways to Make Triangles

Question 9. How many ways can you break a square into triangles by drawing lines between some of the the vertices (without letting any of the lines you draw cross)? Draw every possibility.

Question 10. How many ways can you break a pentagon into triangles by drawing lines between some of the the vertices (without letting any of the lines you draw cross)? Draw every possibility.

Question 11. How many ways can you break a hexagon into triangles by drawing lines between some of the the vertices (without letting any of the lines you draw cross)? Draw every possibility.

Putting it all Together

Question 12. Did you notice any similarities in the numbers you got? What did you notice?

Question 13. We'll call C_n the number of ways we can write n pairs of parentheses. Fill out the chart (C_0 and C_1 are done for you):

$$C_0 = 1 \quad \text{List: } \{ \}$$
$$C_1 = 1 \quad \text{List: } \{ () \}$$
$$C_2 = \text{List: } \{ \text{ }(), () \}$$
$$C_3 = \text{List: } \{ \quad \quad \quad \}$$
$$C_4 = \text{List: } \{ \quad \quad \quad \}$$

Question 14. We can make sets of of 3 pairs of parentheses from smaller sets of parentheses. For example, we can consider sets of the form $(A)B$ where we have:

- 1) A has no pairs, B has 2 pairs
- 2) A has 1 pair, B has 1 pair
- 3) A has 2 pairs, B has no pairs

Use the cases above and the numbers C_0, C_1, C_2 to write a formula for C_3 and calculate it to make sure it is correct. Can you guess a formula for C_n if you have C_0, \dots, C_{n-1} ?

Question 15. See if you can explain why any of our other examples should be defined by the same formula.