UIC Quantum Topology seminar notes

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1 Topology

- The objects in **Diff** are *smooth manifolds*. The arrows are *smooth maps*, and the equivalence arrows are *diffeomorphisms* (bijective smooth maps with smooth inverses). We work primarily with *oriented* manifolds.
- An *embedding* is a smooth map that is a diffeomorphism onto its image.
- A *homotopy* is a continuous family of maps $\{f_t : X \to Y\}$, where $t \in [0,1]$. The maps f_0 and f_1 are the *beginning* and *end* of the homotopy. A sequence of homotopies can be *chained* together if the end of each homotopy equals the beginning of the next.
- An *ambient isotopy* of a manifold Y is a homotopy of self-diffeomorphisms $\{\varphi_t : Y \to Y\}$ where $\varphi_0 = \mathrm{id}_X$. More generally, a homotopy of embeddings $\{\varphi_t : X \to Y\}$ is called an *isotopy.*
- **Theorem:** A self-diffeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is orientation-preserving if and only if it is the end of an ambient isotopy $\{\varphi_t\}$.
- If $\{f_t : X \to Y\}$ is a homotopy, then the map $F : X \times [0,1] \to Y \times [0,1]$ given by $F(x,t) = \big(f_t(x), t\big)$ is the *extrusion* of $\{f_t\}$. (This term is not standard.)

2 Knot diagrams

- A curve $D : S^1 \to \mathbb{R}^2$ is *generic* if:

  I. it is regular, i.e., it has nonvanishing first derivative,

  II. its self-intersections are transverse, and

  III. its self-intersections are of order 2.

  Note that the image of a generic curve is a finite 4-regular plane graph.

- Suppose D is a generic curve in $\mathbb{R}^2$, and θ, θ' are distinct points of S^1 with $D(\theta) = D(\theta')$. An instance of *crossing data* is a formal designation of θ as "over" and θ' as "under", or vice-versa. This is indicated graphically in the image of D by drawing a small break in the understrand around the crossing. If the curve D is regarded as a plane graph, then crossing data is a designation of "over" and "under" strands around a graph vertex.

- A generic curve D with crossing data designated at each crossing is a *knot diagram*. Technically, the diagram is a pair (D, c), where $D : S^1 \to \mathbb{R}^2$ is a curve and c is a set of crossing data. However, we will frequently abbreviate this and simply denote the diagram by D. Alternatively, the knot diagram may be regarded as a plane graph with crossing data at each vertex.

- Two diagrams D_1, D_2 are *isotopic* if there exist orientation-preserving diffeomorphisms $\varphi : S^1 \to S^1$ and $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $D_2 = (\psi \circ D_1 \circ \varphi)$, and crossing data is preserved in the following sense: If $D_1(\theta) = D_1(\theta')$ is a crossing with θ "over" and θ' "under", then $D_2(\varphi\theta) = D_2(\varphi\theta')$ must be a crossing with the same sense. Alternatively, regarding the diagrams as plane graphs, the diagrams are *isotopic* if there is an isomorphism of the plane graphs preserving crossing data.

3 Knots and projections

- A *knot* is an embedding $k : S^1 \to \mathbb{R}^3$.

- Two knots k, k' are *equivalent* (or *ambiently isotopic*) if there is an orientation-preserving diffeomorphism $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ with $k' = \varphi \circ k$.

- The *standard z-axis projection* of $\mathbb{R}^3$ is the submersion $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ given by $\pi(x, y, z) = (x, y)$.

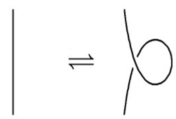
- A knot k is in *generic position* if the composition $D = (\pi \circ k)$ is a generic curve.

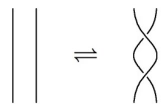
- Given a knot k in generic position, apply crossing data to the generic curve $D = (\pi \circ k)$ in the following way: For each crossing $D(\theta) = D(\theta')$, let θ be "over" if the z-coordinate of $k(\theta)$ is greater than that of $k(\theta')$. The result is a knot diagram, called *the diagram of k*.

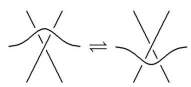
- **Theorem:** Every knot k is equivalent to a knot k' in generic position. In fact, for any $\varepsilon > 0$, there is a diffeomorphism $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ with $|\varphi(x) - x| < \varepsilon$ for all $x \in \mathbb{R}^3$, such that $k' = \varphi k$ is in generic position.

4 Diagrammatic equivalence

- A *Reidemeister move* is a transformation of a diagram in a local region, leaving all else unchanged, of one of three types. This may be interpreted as a transformation of either a generic curve with crossing data, or a plane graph with crossing data. The three moves are:

I. 

II. 

III. 

Note that isotopic diagrams are regarded as equal, so only the plane graph structure and crossing data are relevant.

- Two diagrams D, D' are considered *equivalent* if there is a sequence of diagrams
$$D = D_1 \sim \cdots \sim D_n = D'$$
where each D_i is related to D_{i+1} by a Reidemeister move.

- When crossing data is disregarded, we may refer to these moves as *flat* Reidemeister moves. The ordinary Reidemeister moves are formed by adding crossing data to the three flat moves as shown above. Any application of crossing data to the flat moves other than those listed above is a *forbidden move.*

- Suppose D is related to D' by a single Reidemeister move. Regard the diagrams as generic curves $S^1 \to \mathbb{R}^2$. There is a homotopy $\{D_t\}$ (where $t \in [0,1]$) with $D_0 = D$ and $D_1 = D'$, where D_t is isotopic to D for $t < 0.5$ and isotopic to D' for $t > 0.5$. The curve $D_{0.5}$ fails to be generic at a single point of the image:

  I. For an R-I move, $D_{0.5}$ has a (non-crossing) point where the derivative vanishes.
  II. For an R-II move, $D_{0.5}$ has a point of order-2 non-transverse self-intersection.
  III. For an R-III move, $D_{0.5}$ has a triple transverse intersection.

- If D and D' are equivalent, there is a homotopy "through Reidemeister moves" from D to D', formed by chaining together the homotopies for each Reidemeister move in the sequence relating D to D'.

- Suppose k and k' are equivalent knots in generic position, and φ is an orientation-preserving diffeomorphism with $k' = \varphi \circ k$. Then the diagrams $D = \pi \circ k$ and $D' = \pi \circ k$ are equivalent, and there is an ambient isotopy $\{\varphi_t\}$ of $\mathbb{R}^3$ such that $\{\pi \circ \varphi_t \circ k\}$ is a homotopy through Reidemeister moves from D to D'.

5 Moves, extruded

- If $\{D_t : S^1 \to \mathbb{R}^2\}$ is a curve moving around in the plane, then its extrusion $E : S^1 \times I \to \mathbb{R}^2 \times I$ is a surface in 3-space. A generic crossing of D_t becomes a point on a double-point arc of E, that is, a curve where two parts of the surface cross.

- The homotopies corresponding to the three flat Reidemeister moves extrude to the following *singularity forms* of a surface in $\mathbb{R}^3$:

  I. A *Whitney umbrella branch point*

  II. A local t-max or t-min of a double-point arc

  III. A standard triple point.

- Suppose D_0 and D_1 are generic curves, and $\{D_t\}$ is a homotopy through (flat) Reidemeister moves. Then the extrusion E is a surface in 3-space whose singular set consists of the three forms just listed. Such a surface is called *generic*.