Symmetry Groups of Knot Embeddings

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The purpose of this study is to associate knot embeddings and point groups of \( \mathbb{R}^3 \).

Point groups consist of isometries of \( \mathbb{R}^3 \) that fix the origin—specifically, rotations, reflections, and rotoreflections. The symmetry of an object \( X \) in \( \mathbb{R}^3 \) is described by the largest point group whose elements map \( X \) to itself, called the symmetry group of \( X \). For example, a cube has six axes of twofold rotation, four axes of threefold rotation, and three axes of fourfold rotation, as well as nine planes of mirror reflection; these symmetries each correspond to elements of the point group \( O_h \), and these elements generate the group. \( O_h \) has order 48, because there are 48 isometries of \( \mathbb{R}^3 \) that map the cube to itself.

Knot embeddings, in particular, are objects in \( \mathbb{R}^3 \), so their geometric symmetries are described by point groups. Note that these symmetries are not invariants of the abstract knot itself, but rather characterize a given embedding of the knot in \( \mathbb{R}^3 \). For example, a trefoil knot can be arranged in space so that its point group is \( D_3, D_2, C_3, C_2, \) or \( C_1 = \{ \text{Id} \} \) the trivial group. See figure 1. Note that every knot has an embedding with trivial point group, that is, an asymmetrical embedding.

We have two goals in mind: First, to find a knot or link exemplifying every discrete point group. Second, to find for certain knots and links (or classes thereof or, in the long run, all knots and links) all possible symmetry groups for embeddings of said knots. For example, the five groups \( D_3, D_2, C_3, C_2, \) & \( C_1 \) mentioned in the preceding paragraph comprise a complete listing of all possible symmetry groups for embeddings of a trefoil.

First we present a brief overview of point groups in \( \mathbb{R}^3 \). Point groups fall into two distinct categories: Infinite and discrete.

Examples of infinite point groups include \( O(3) \), the group of all sphere isometries, called the Sphere Group or the orthogonal group of dimension 3. All 3-D point groups are, by definition, subgroups of \( O(3) \). Another example is the group of arbitrary rotations about a single axis, reflections in all planes containing that axis, reflections in a single plane perpendicular to that axis, and 2-fold rotations about radial axes in this perpendicular plane. This is the symmetry group of a right circular cylinder of finite height; we call this the Cylinder Group or \( D_{\infty h} \), isomorphic to \( \mathbb{R}/\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \). One important infinite subgroup of the Cylinder Group is the group of arbitrary rotations about a single axis and reflections in planes containing that axis; this is the symmetry group of a single cone, so we call it the Cone Group or \( C_{\infty v} \), isomorphic to \( \mathbb{R}/\mathbb{Z} \oplus \mathbb{Z}_2 \).

PROPOSITION: The only knot embeddings with infinite symmetry are the unknot embedded as a perfect circle, whose symmetry group is the Cylinder Group. The only link embeddings with infinite symmetry are trivial links with all components perfect circles lying in parallel planes and having a common axis of rotation. If the embedding has a plane of reflection parallel to the planes of the circles, then the symmetry group
is the Cylinder Group; otherwise, the symmetry group is the Cone Group. See figure 2. Embeddings of non-trivial knots and links never have infinite symmetry groups.

Because of the preceding proposition, we shall chiefly be interested in discrete point groups. These fall into two basic types: The uniaxial groups, which are subgroups of the Cylinder Group; and the polyaxial groups, which are not. Though there are infinitely many uniaxial groups, there are exactly seven polyaxial groups, to be described below.

Uniaxial groups are so called because they have at most one axis of $n$-fold rotation with $n > 2$. In terms of abstract group theory, that is to say the group contains at most one cycle of length $> 2$. As a subgroup of the cylinder group, this $n$-fold axis is the cylinder’s axis.

The seven polyaxial groups are characterized by at least four 3-fold axes, aligned with the vectors $(1,1,1)$, $(-1,1,1)$, $(1,-1,1)$, & $(1,1,-1)$ in $\mathbb{R}^3$. Some of these groups have additional 3-fold axes as well as 4-fold and/or 5-fold axes, as we shall see.

It should be noted that objects in $\mathbb{R}^3$ do not have to be centered at the origin and aligned along specific axes to be associated with a particular point group. By conjugating the group’s elements with isometries that move the object into the desired position, we can align the object’s rotation axes and reflection planes with those corresponding to the preconceived elements of the point group. For example, a cube in arbitrary position in $\mathbb{R}^3$ is still considered to have point group $O_h$, even though the elements of $O_h$ are isometries that fix the origin. Therefore, we will identify point groups in $\mathbb{R}^3$ “up to conjugacy”, meaning they represent the symmetries of an object in possibly different positions in space.

Point groups are identified by the specific isometries of $\mathbb{R}^3$ they contain, rather than by their abstract group type. In fact, it is easy to find point groups that are isomorphic as abstract groups, but represent entirely different sets of isometries of $\mathbb{R}^3$, even up to conjugacy. For example, the point group $C_2$ contains the identity element and a $180^\circ$ rotation about an axis, so it is isomorphic to the abstract group $\mathbb{Z}_2$. The point group $S_2$ contains the identity element and the inversion isometry (which maps every point $x$ to $-x$), so it is also isomorphic to $\mathbb{Z}_2$. And the point group $C_{1h}$, containing only the identity element and the reflection in a single plane, is isomorphic to $\mathbb{Z}_2$. However, $C_2$, $S_2$ and $C_{1h}$ are distinct as point groups because they are not conjugate; that is, there is no isometry $g$ such that $C_2 = g^{-1}S_2g$, or $C_2 = g^{-1}C_{1h}g$, or $S_2 = g^{-1}C_{1h}g$. For the first two cases, it suffices to note that $C_2$ preserves the orientation of $\mathbb{R}^3$ whereas $S_2$ and $C_{1h}$ do not; conjugation cannot reverse the orientation of $\mathbb{R}^3$. For the third case, note that elephants have flat feet.

A subgroup of a point group is also a point group. In the trefoil example, notice that all subgroups of each of the five groups listed are part of the list. Specifically, $D_3 \subset C_3 \subset C_1$ and $D_2 \subset C_2 \subset C_1$, and none of the groups have any subgroups other than these. This situation is a case of the following general fact.

PROPOSITION: If a given knot has an embedding whose symmetry group is $G$, then for every subgroup $H \subseteq G$ there is an embedding of the same knot whose symmetry group is $H$. For this reason, in describing the complete set of possible symmetry groups of a knot, it suffices to name only those point groups which are maximal, that is, which are not proper subsets of any possible symmetry group of the knot. For example, a trefoil’s possible symmetries are the subgroups of $D_3$ and $D_2$.

Let’s look at some examples of point groups, and illustrate them with knot embeddings. The uniaxial groups are classified into seven infinite series. There are listed with their order and a description of the symmetry they represent. The wikipedia graphic is good; it is figure 3.
- $C_n$: The cyclic group of order $n$. Generated by a single $n$-fold rotation axis (the cylinder axis).
- $C_{nh}$: Order $2n$. Generated by the $n$-fold cylinder axis, plus a reflection plane perpendicular to that axis.
- $C_{nv}$: Order $2n$. Generated by the $n$-fold cylinder axis, plus $n$ evenly-spaced reflection planes containing that axis. Note that $C_{1v} = C_{1h}$ (where equality is up to conjugacy).
- $D_n$: The dihedral group of order $2n$. Generated by the $n$-fold cylinder axis, plus $n$ evenly-spaced 2-fold rotation axes perpendicular to the cylinder axis. Note that $D_1 = C_2$.
- $D_{nh}$: Order $4n$. Generated by the dihedral group, plus a reflection plane perpendicular to the cylinder axis, plus $n$ reflection planes containing the cylinder axis and one of the 2-fold axes.
- $D_{nd}$: Order $4n$. Generated by the dihedral group, plus $n$ reflection planes containing the cylinder axis but not any other axes. Note that $D_{1d} = C_{2h}$.
- $S_n$: The $n$-fold inversion group of order $n$. Consists of one $n$-fold rotoreflection—that is, $360°/n$ rotation about one axis combined with reflection in a perpendicular plane. Note that $S_1 = C_1 = \{\text{Id}\}$. The point group $S_2$ is the inversion symmetry group mentioned earlier; a parallelepiped with unequal sides, for example, has this type of symmetry.

For each of the uniaxial point groups there is an embedding of the unknot with that group as its symmetry group. See figure 4 for examples of each of the seven types. The same symmetry can be achieved in trivial links of $n$ components—just take $n$ duplicates of the unknot embedding with the desired symmetry and position them directly over one another along the cylinder axis.

PROPOSITION: An $(m, n)$ torus knot or link has embeddings with symmetry groups $D_m$ and $D_n$. It also has embeddings with symmetry groups equal to any subgroup of $D_m$ and $D_n$, that is $D_p$ and $C_p$ for all integers $p$ dividing $m$ or $n$. See figure 5 for examples.

Of course, there are knots and links that have the same set of possible symmetries as a torus knot, but are not torus knots themselves; see figure 6.

Mirror symmetry occurs in knots and links that are composites of (or separable unions of) mirror images. In this way the point groups $D_{nd}$ can be realized as symmetry groups of non-trivial knots. For example, the composite of two trefoil knots which are mirrors of each other can be given $D_{3d}$ symmetry; see figure 7.

Let us now move on to the polyaxial point groups, which have at least four 3-fold rotation axes in their set of generators. There are exactly seven such point groups.

- $T$: Chiral tetrahedral symmetry, of order 12. Generated by four 3-fold axes (aligned with the vectors $(1, 1, 1), (1, -1, 1), (1, 1, -1)$, & $(1, -1, -1)$ in $\mathbb{R}^3$, called the “tetrahedron axes”); plus three 2-fold axes (aligned with the $x$, $y$, and $z$ axes, called the "cartesian axes"). This group is isomorphic to $A_4$ (the alternating group on four letters).
- $T_d$: Full tetrahedral symmetry, of order 24. Generated by the tetrahedron and cartesian axes, plus six mirror planes (each containing one tetrahedron axis and one cartesian axis). This group is isomorphic to $S_4$ (the symmetric group on four letters).
• $T_h$: Pyritohedral symmetry, of order 24. Generated by the tetrahedron and cartesian axes, plus three mirror planes (corresponding to the $xy$, $yz$, and $xz$ planes, called the "cartesian planes"). This group is isomorphic to $A_4 \times C_2$.

• $O$: Chiral octahedral symmetry, of order 24. The tetrahedral and cartesian axes are retained but the cartesian axes are now 4-fold instead of 2-fold. Also there are six 2-fold axes (corresponding to the vectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(0,1,1)$, $(1,0,1)$, & $(1,1,0)$). This group is isomorphic to $S_4$, the same as $T_d$.

• $O_h$: Full octahedral symmetry, of order 48. Generated by the same thirteen axes as $O$, plus thirteen planes of reflection, each one perpendicular to a rotation axis. This group is isomorphic to $S_4 \times C_2$.

• $I$: Chiral icosahedral symmetry, of order 60. Generated by ten 3-fold axes (including the tetrahedral axes), fifteen 2-fold axes (including the cartesian axes), and six 5-fold axes (called the "dodecahedral axes"). This group is isomorphic to $A_5$.

• $I_h$: Full icosahedral symmetry, of order 120. Generated by the same 31 axes as $I$, plus fifteen reflection planes, each one perpendicular to a 2-fold axis. This group is isomorphic to $A_5 \times C_2$.

PROPOSITION: The only link embedding having symmetry group $T_d$ is the trivial link of $K$ components, where $K = 4f + 6e + 4v$ for non-negative integers $f$, $e$, and $v$, not all zero. The only link embedding having symmetry group $O_h$ is the trivial link of $K$ components, where $K = 8f + 12e + 6v$. The only link embedding having symmetry group $I_h$ is the trivial link of $K$ components, where $K = 20f + 30e + 12v$.

PROPOSITION: A link embedding with symmetry group $T$ or $T_h$ need not be trivial, and has $K$ components, where $K = 4f + 3e + 4v$. (Example: Borromean rings, \textit{figure 8}) A link embedding with symmetry group $O$ need not be trivial, and has $K$ components, where $K = 4f + 12e + 6v$. (See \textit{figure 8} again.) A link embedding with symmetry group $I$ need not be trivial, and has $K$ components, where $K = 20f + 30e + 6v$. (Example: A Nobbly-Wobbly. See \textit{figure 9}.)

These propositions tell us that no knot or link with fewer than 3 components can have polyaxial symmetry. Specifically, 3 are needed for chiral tetrahedral or pyritohedral; 4 are needed for chiral octahedral; and 6 are needed for chiral icosahedral. For trivial links, 4 are needed for full tetrahedral; 6 are needed for full octahedral; and 12 are needed for full icosahedral.

Many knots and links have no embeddings with non-trivial symmetry. Lou Kauffman has suggested the term \textit{irascible}, meaning hot-headed or anger-prone, to describe such knots and links; however I prefer the duller term \textit{non-symmetrizable}, reasoning that no matter how asymmetrical they may be, knots and links are by nature peaceful level-headed entities, particularly the rational knots.

I will add information here about knots that touch the point at infinity, and the relationship between mirror symmetry and reversible knots.