Notes on the geometry of curves, Math 210

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Basic definitions.

Let $f(t)$ be a vector-valued function of a scalar. We indicate this by writing $f : R \to R^3$ and think of $f(t)$ as the position in space of a particle at time t; f defines the orbit of the particle.

$f(t+h) - f(t)$ is the displacement in space during the time interval from t to $t+h$.

$(1/h)(f(t+h) - f(t))$ is the [displacement]/[time interval] or average velocity.

$f'(t) = \lim_{h \to 0} \frac{1}{h} \left(f(t+h) - f(t) \right)$ is the velocity.

$||f'(t)||$ is the speed.

$T(t) = \frac{1}{||f'(t)||} f'(t)$ is the direction vector or unit tangent vector. It is defined whenever $f'(t) \neq \vec{0}$.

$f'(t) = ||f'(t)||T(t)$ expresses velocity as speed (a scalar) times direction (a vector).

$s(t) = \int_{t_0}^{t_1} ||f'(t)||\, dt$ is the distance traveled along the orbit during the time interval from t_0 to t_1, or the arclength of the curve.

$f''(t)$ is the acceleration.

An example. The helix is defined by $f(t) = (a\cos\omega t, a\sin\omega t, bt)$.

$f'(t) = (-a\omega \sin\omega t, a\omega\cos\omega t, b)$.

$||f'(t)|| = \sqrt{a^2\omega^2 + b^2}$.

$T(t) = \frac{1}{\sqrt{a^2\omega^2+b^2}}(-a\omega\sin\omega t, a\omega\cos\omega t, b)$.

The distance along the helix from $f(0)$ to $f(2\pi/\omega)$ is

$$\int_0^{2\pi/\omega} \sqrt{a^2\omega^2 + b^2}\, dt = (2\pi/\omega)\sqrt{a^2\omega^2+b^2}.$$

The displacement is $f(2\pi/\omega) - f(0) = (a, 0, 2\pi b/\omega) - (a, 0, 0) = (0, 0, 2\pi b/\omega)$.

The acceleration is $f''(t) = (-a\omega^2\cos\omega t, -a\omega^2\sin\omega t, 0)$.

Notice that $f''(t) = f'(t) \times (0, 0, -\omega)$. For any orbit f that satisfies this equation the acceleration is perpendicular to the velocity, so the speed will be constant, as it is for our helix. The force on a particle with electric charge q moving with velocity v in a magnetic field B is

$$F = qv \times B.$$

The acceleration due to this force is given by

$$F = mf''(t)$$

where m is the mass of the particle. So the motion of the particle satisfies

$$mf''(t) = qf'(t) \times B.$$

Our helix will satisfy this equation if $B = -\frac{\omega m}{q}\vec{k}$. The frequency $\frac{\omega}{2\pi} = \frac{1}{2\pi}\frac{q}{m}||B||$ is independent of the speed (provided the speed is low enough that the mass can be regarded as constant). This fact played a role in the design of the cyclotron particle accelerator by Lawrence and Livingston in 1931.

Curves. To study properties of the curve itself rather than properties of motion along the curve, we consider the curve parameterized by arclength. If $c(s)$ is a vector function of a parameter s, the condition that s be arclength is that the speed $||c'(s)|| = 1$. Then $T(s) = c'(s)$ is the unit tangent vector.

Since $T(s) \cdot T(s) = 1$, differentiating we find $T'(s) \cdot T(s) = 0$, so $T'(s) \perp T(s)$.

We define the curvature of c to be $k(s) = ||T'(s)||$.

If $k(s) \neq 0$, define the principal normal vector by

$$N(s) = \frac{1}{k(s)}T'(s).$$

Then $T'(s) = k(s)N(s)$. The vectors $T(s)$ and $N(s)$ are perpendicular to each other and of length one. The binormal vector defined by

$$B(s) = T(s) \times N(s)$$

has length one and is perpendicular to both T and N. The triple T, N, and B is an orthonormal basis of R^3 and can be obtained from the standard orthonormal basis $\vec{i}$, $\vec{j}$, and $\vec{k}$ by a rotation of 3-space. Just as with the standard basis, an arbitrary vector can be written as a sum in the form

$$V = aT + bN + cB.$$

To find the coefficients a, b, and c, use the dot product: $V \cdot T = a$, $V \cdot N = b$, and $V \cdot B = c$.

Since $N(s) \cdot N(s) = 1$, $N'(s) \perp N(s)$, and since $N(s) \cdot T(s) = 0$ we have

$$N'(s) \cdot T(s) = -N(s) \cdot T'(s) = -k(s).$$

We define the torsion of c to be $\tau(s) = N'(s) \cdot B(s)$. Then $N'(s) = -k(s)T(s) + \tau(s)B(s)$. Finally by differentiation we find $B' = T' \times N + T \times N' = kN \times N + T \times (-kT + \tau B) = -\tau N$.

At each point $c(s)$ along the curve c we have a unit vectors T tangent to the curve, N perpendicular to T and pointing in the direction that T is turning, and B completing an orthonormal basis. The rotation of these vectors as we move along the curve is described by the Frenet-Serret formulas for their derivatives:

$$\begin{aligned} T' &= \quad\ \ kN \\ N' &= -kT \qquad \tau B \\ B' &= \quad -\tau N \end{aligned}$$

If we specify a given point $\vec{p}$, direction $\vec{u}$, and functions, $k(s)$ (nonvanishing) and $\tau(s)$ which have third derivatives, then a fundamental theorem of differential equations implies the existence of a curve $c(s)$ with $c(0) = \vec{p}$, $c'(0) = \vec{u}$, $||c'(s)|| = 1$, and with curvature and torsion equal to the given function k and τ. Furthermore the curve is uniquely determined by the initial conditions $c(0) = \vec{p}$ and $c'(0) = \vec{u}$ and the curvature and torsion functions.

Example. The helix above with $\omega = 1$ has constant speed $\sqrt{a^2 + b^2}$. Replacing t by $s/\sqrt{a^2 + b^2}$ we get

$$c(s) = \left(a \cos \frac{s}{\sqrt{a^2+b^2}},\ a \sin \frac{s}{\sqrt{a^2+b^2}},\ \frac{bs}{\sqrt{a^2+b^2}}\right).$$

$$c'(s) = T(s) = \frac{1}{\sqrt{a^2+b^2}} \left(a \sin \frac{s}{\sqrt{a^2+b^2}},\ a \cos \frac{s}{\sqrt{a^2+b^2}},\ b\right).$$

$$c''(s) = T'(s) = \frac{a}{a^2+b^2} \left(-\cos \frac{s}{\sqrt{a^2+b^2}},\ -\sin \frac{s}{\sqrt{a^2+b^2}},\ 0\right).$$

$$\text{So} \quad k(s) = \frac{a}{a^2+b^2} \quad \text{and} \quad N(s) = \left(-\cos \frac{s}{\sqrt{a^2+b^2}},\ -\sin \frac{s}{\sqrt{a^2+b^2}},\ 0\right).$$

$$N'(s) = \frac{1}{\sqrt{a^2+b^2}} \left(\sin \frac{s}{\sqrt{a^2+b^2}},\ -\cos \frac{s}{\sqrt{a^2+b^2}},\ 0\right).$$

$$B = T \times N = \frac{1}{\sqrt{a^2+b^2}} \left(b \sin \frac{s}{\sqrt{a^2+b^2}},\ -b \cos \frac{s}{\sqrt{a^2+b^2}},\ a\right).$$

$$\tau(s) = N' \cdot B = \frac{b}{a^2+b^2}.$$

Motion along a curve. Suppose a particle follows the curve $c(s)$ at varying speed. Its distance along the curve will be given by some function $s(t)$. Its position is given by $f(t) = c(s(t))$.

$$\text{The velocity is } f'(t) = c'(s(t))s'(t) = s'(t)T(s(t)), \text{the speed is } ||f'(t)|| \ = s'(t),$$

$$\begin{aligned} \text{and the acceleration is } f''(t) &= s''(t)T(s(t)) + (s'(t))^2 T'(s(t)) \\ &= s''(t)T(s(t)) + k(s(t))(s'(t))^2 N(s(t)). \end{aligned}$$

The tangential component of acceleration is the rate of change of speed, the normal component is the curvature times the square of the speed.

These formulas for velocity and acceleration can be used to give a formula for curvature when the parameter is not necessarily arclength.

$$\begin{aligned} f''(t) \times f'(t) &= \left(s''T + ks'^2N\right) \times s'T \\ &= ks'^3 N \times T = -ks'^3 B. \end{aligned}$$

Hence $k(t) = \dfrac{|f''(t) \times f'(t)|}{|f'(t)|^3}$.

Taylor series. We expand $c(s)$ in a Taylor series around $s = 0$. The derivatives are given by:

$$\begin{aligned} c'(s) &= T(s) \\ c''(s) &= T'(s) = k(s)N(s) \\ c'''(s) &= k'(s)N(s) + k(s)N'(s) \\ &= -k^2T + k'N + k\tau B. \end{aligned}$$

The expansion of $c(s)$ is:

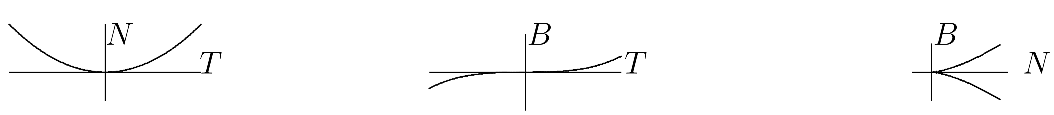
$$\begin{aligned} c(s) = c(0) + sT(0) &+ \frac{1}{2}s^2k(0)N(0) \\ &+ \frac{1}{6}s^3\left(-k(0)^2T(0) + k'(0)N(0) + k(0)\tau(0)B(0)\right) + \cdots . \end{aligned}$$

The leading term behavior for each component is

$$c(s) \approx c(0) + sT(0) + \frac{1}{2}s^2k(0)N(0) + \frac{1}{6}s^3k(0)\tau(0)B(0).$$

The projection of this approximation onto the plane spanned by T and N (the osculating plane) is $(s, s^2k(0)/2)$. The projection onto the plane spanned by T and B (the rectifying plane) is $(s, s^3k(0)\tau(0)/6)$. The projection onto the plane spanned by N and B (the normal plane) is $(k(0)/2)(s^2, s^3\tau(0)/3)$.

Views of a curve



Osculation plane — Rectifying plane — Normal plane

An activity. Find a clear cylindrical drinking glass and some narrow colored tape. If you place the tape on the glass starting down at an angle from the top it will follow a helix. Be careful not to bend the tape to the side which will cause it to pucker and deviate from a helix. Visualize the vectors T, N, and B at some point on the helix. Then view the curve by looking straight down first B, then N, and then T. What you see should look like the views of a curve above.

Geometric consequences of differential assumptions.

If $\tau = 0$, c is a plane curve.

Proof. Since $B'(s) = -\tau(s)N(s)$, we have $B'(s) = 0$, so $B(s)$ is a constant vector, B. We will show that the curve lies in a plane perpendicular to B, that is that $c(s) - c(0) \perp B$. The derivative of $(c(s) - c(0)) \cdot B$ is $c'(s) \cdot B = T \cdot B = 0$, so this dot product is a constant. Taking $s = 0$ we see that $c(s) - c(0) \perp B$.

Exercise: If $k = 0$, c lies on a straight line.

If $\tau = 0$ and $k(s)$ is constant, then $c(s)$ lies on a circle.

Proof. The curvature of a circle of radius a is $1/a$. The center is at $c(s) + aN(s)$. Hence with our hypotheses,

$$A(s) = c(s) + \frac{1}{k(s)}N(s)$$

should be a constant vector. Differentiate $A(s)$ and use the Frenet-Serret formulas to conclude that $A(s)$ is constant, say equal to A. Then $||c(s) - A|| = 1/k$ is constant and $(c(s) - A) \cdot B = 0$. Therefore $c(s)$ lies on the circle of radius $1/k$ about $c(0) + (1/k)N(0)$ in the plane perpendicular to B through that center point.

Exercise: If k and τ are constant, c is a helix.

Curves on a sphere. Suppose $c(s)$ lies on the sphere of radius r. We know that a straight line is not curved enough to lie on a sphere. What is the least curved curve that lies on the sphere? The condition that $c(s)$ lie on the sphere is $c(s) \cdot c(s) = r^2$. Differentiating we find $c \cdot c' = 0$ and $c' \cdot c' + c \cdot c'' = 0$. Since we assume c is parameterized by arclength, $c' \cdot c' = 1$. So $c \cdot c'' = -c' \cdot c' = -1$ and therefore $kc \cdot N = -1$. If θ is the angle between c and N, $c \cdot N = ||c||\,||N|| \cos\theta$. Therefore $|c \cdot N| \leq ||c||\,||N|| = r$, so $k \geq 1/r$. This shows $1/r$ is the least possible curvature at any point along a curve on the sphere of radius r.

If $k = 1/r$, then $c \cdot N = -r$, $\theta = \pi$, and $N = -(1/r)c$. Then $N' = -(1/r)c' = -(1/r)T$. By the Frenet-Serret formulas this implies that $\tau = 0$. Therefore B is constant and $c \cdot B = -rN \cdot B = 0$, so c lies in the plane through the origin perpendicular to B which meets the sphere in a great circle.

The least curved curve which lies on a sphere is a great circle.

Sweeping area. Consider the orbit $f(t)$ of a particle and two nearby points $f(t)$ and $f(t + h)$. The area swept out by the line from the origin to the orbit as time goes from t to $t + h$ is approximately half the area of the parallelogram spanned by $f(t)$ and $f(t + h)$.

$$\text{Area}(h) \approx \frac{1}{2}||f(t) \times f(t + h)||.$$

$$\begin{aligned}
\text{Rate of sweep of area} &= \lim_{h\to 0^+} \frac{\text{Area}(h)}{h} \\
&= \frac{1}{2} \lim_{h\to 0^+} \frac{||f(t) \times f(t+h)||}{h} \\
&= \frac{1}{2} \lim_{h\to 0} ||f(t) \times \frac{(f(t+h) - f(t))}{h}|| \\
&= \frac{1}{2}||f(t) \times f'(t)||.
\end{aligned}$$

A central force is a force which, at each point, is directed along the line toward (or away from) the origin, $F(\vec{v})$ is a scalar function times $\vec{v}$. It follows that $\vec{v} \times F(\vec{v}) = \vec{0}$. For example the gravitational force is given by

$$F(\vec{v}) = -\frac{GmM}{||\vec{v}||^2} \frac{v}{||v||}.$$

If a particle of mass m is moving in an orbit $f(t)$ according to Newton's law

$$mf''(t) = F(f(t)),$$

then $f(t) \times f''(t) = 0$. Differentiating $(f \times f')' = f' \times f' + f \times f'' = 0$. Hence $f \times f' = A$ is a constant vector. Therefore the rate of sweep of area is constant, $f(t)$ sweeps out equal areas in equal times. A is perpendicular to f and f' and also to f'' because it is a multiple of f by the central force assumption. Hence $A \perp T$ and $A \perp N$, so A is a multiple of B and B is constant. Therefore the motion lies in a plane. This argument derives Kepler's second law from Newton's dynamics.

Exercise: Let $x(t)\vec{i} + y(t)\vec{j}$, $a \leq t \leq b$, be a curve in the plane which encloses a region R. Use the rate of sweep of area formula to show that the area of R is given by

$$\frac{1}{2} \int_a^b x(t)y'(t) - x'(t)y(t)\, dt.$$