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Differentiable Manifolds—Problems

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The goal of problems 1 – 5 is stated in problem 5. They appear in *Differential Topology* by M. W. Hirsch and are taken from a 1940 paper by Heinz Hopf. The problems concern maps given in terms of a symmetric bilinear map $\beta : R^{n+1} \times R^{n+1} \longrightarrow R^{2n+1}$ where $\beta(x, y) = z$, $x = (x_0, \ldots, x_n)$, $y = (y_0, \ldots, y_n)$, and where $z = (z_0, \ldots, z_{2n})$ is given by

$$z_k = \sum_{i+j=k} x_i y_j.$$

1. Show if $\beta(x, y) = 0$ then $x = 0$ or $y = 0$.

2. Define $g : S^n \longrightarrow S^{2n}$ by $g(x) = \beta(x, x)/|\beta(x, x)|$. Show
   (a) if $g(x) = g(y)$ then $\beta(x, x) = t^2\beta(y, y)$ for some $t \neq 0$,
   (b) that $\beta(x + ty, x - ty) = 0$, and hence
   (c) $x = \pm y$.

3. Define $f : RP^n \longrightarrow S^{2n}$ by $f([x]) = g(x)$. Show f is one-to-one. Conclude that f is a topological embedding.

4. Show f is C^∞.

5. Show the point $p = (0, 1, 0, \ldots, 0) \in S^{2n} - \operatorname{im} f$ and hence the stereographic projection φ from p gives an embedding $\varphi \circ f : RP^n \longrightarrow R^{2n}$.

6. Define $f : M_n(R) \longrightarrow M_n(R)$ by $f(X) = XX^t$ where X is an $n \times n$-matrix and X^t is its transpose.
   (a) $f(X)$ is symmetric,
   (b) $Df(X)M = XM^t + MX^t$,
   (c) If $X \in O(n)$, that is, if $f(X) = I$, then $Df(X)$ maps $M_n(R)$ onto the space of symmetric matrices. Hint: Given a symmetric S, take $M = \frac{1}{2}SX$.

7. Recall from §5, for a linear map ℓ, $||\ell|| = \sup\{|\ell(x)|/|x| : x \neq 0\}$.
   Show $||\lambda \circ \ell|| \leq ||\lambda||\ ||\ell||$.

8. Let $G\ell_n(R) \subset M_n(R)$ be the open subset of invertible matrices; $X \in G\ell_n(R)$ iff $\det(X) \neq 0$. Let $f(X) = X^{-1}$. Show $Df(X)M = -X^{-1}MX^{-1}$.

9. Let

$$f(x,y) = \begin{cases} x^2y/(x^2+y^2), & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Show that $D_v f(\vec{0})$ exists for all v, but that $Df(\vec{0})$ does not exist since $v \mapsto D_v f(\vec{0})$ is not linear. See Dieudonné 8.4 problem 3(b) for an example where $D_v f(\vec{0})$ is linear in v, but still $Df(\vec{0})$ does not exist.

The next five problems are from M. Hirsch's *Differential Topology*. They may require some thought or a suggested place to start. They are really exercises on the definitions.

10. An injective immersion of a compact Hausdorff manifold is an embedding.

11. There is an immersion of the punctured torus $S^1 \times S^1 - \{\text{point}\}$ in R^2. Suggestion: since a disk minus a point is homeomorphic to a disk minus a smaller disc in its interior, you can consider $S^1 \times S^1$ minus a large homeomorph of the disk. It may help to regard $S^1 \times S^1$ as a quotient space of the square.

12. Any product of spheres can be embedded in R^{m+1} where the product has dimension m. Suggestion: an inductive proof can be based on the statement, if M^m embeds in R^{n+1} with trivial normal bundle, then M can be embedded in R^{m+1+k} with trivial normal bundle, and hence $M \times S^k$ embeds in R^{m+1+k} with trivial normal bundle.

13. If M is a compact C^1 manifold, every C^1 map $M \longrightarrow R$ has at least two critical points.

14. Regarding the tangent bundle to S^n as

$$TS^n = \{(x,v) : x \in S^n, v \in x^\perp \subset R^{n+1}\},$$

construct a smooth map f,

$$\begin{array}{ccc} TS^n \times R & = & \{(x,v,r) : x \in S^n,\ v \in x^\perp,\ r \in R\} \\ f\big\downarrow & & \big\downarrow \\ S^n \times R^{n+1} & = & \{(x,w) : x \in S^n,\ w \in R^{n+1} \end{array}$$

The next 4 problems are from Glen Bredon's, *Topology and Geometry*. Some manifolds are presented as inverse images of regular values. For more on problems 17 and 18 see John Milnor's, *Singular Points of Complex Hypersurfaces*, Annals of Math. Studies 61.

15. Consider the real valued function $f(x, y, z) = (2 - \sqrt{x^2 + y^2})^2 + z^2$ on $R^3 - \{(0, 0, z)\}$. Show that 1 is a regular value of f and identify the manifold $M = f^{-1}(1)$. [Solve $f(\vec{x}) = 1$ and $\operatorname{grad} f(\vec{x}) = \vec{0}$. To identify M, cylindrical coordinates may be helpful.]

16. Show the manifold M of problem 15 is not transverse to the plane

$$N = \{(x, y, z) \in R^3 : x = 1\}.$$

Is $M \cap N$ a manifold? Describe $M \cap N$. [Show $\operatorname{grad} f$ is normal to N at some point $\vec{x}$ of $M \cap N$, so that $T_{\vec{x}}N = T_{\vec{x}}M$.]

17. Let $V = \{(z_1, z_2, z_3) \in C^3 - \{\vec{0}\} : p(z_1, z_2, z_3) = 0\}$ where $p(z_1, z_2, z_3) = z_1^3 + z_2^2 + z_3^2$. Show V is a 4-dimensional manifold by showing that $0 \in C$ is a regular value of $p : C^3 - \{\vec{0}\} \longrightarrow C$. Show $(0, 0, 0)$ is a singular point of p.

18. Let $S^5 = \{(z_1, z_2, z_3) \in C^3 : g(z_1, z_2, z_3) = 1\}$ where $g(z_1, z_2, z_3) = z_1\overline{z_1} + z_2\overline{z_2} + z_3\overline{z_3}$. Show $V \cap S^5$ is a 3-manifold by showing that $(0, 1) \in C \times R$ is a regular value of (p, g). Since, by problem 17, $dp : (T_zV)^\perp \longrightarrow T_{p(z)}$ is onto for $z \in V$, it is enough to find $\vec{v}_z \in T_zV$ with $dg(v_z) \neq 0$ for $z \in V \cap S^5$. Consider the curve $\gamma : (1 - \varepsilon, 1 + \varepsilon) \longrightarrow V$ defined by $\gamma(t) = (t^2 z_1, t^3 z_2, t^3 z_3)$ and check (i) $g \circ \gamma(1) = 1$ and (ii) $(g \circ \gamma)'(1) \neq 0$.

19. For 1-forms ω, the derivative satisfies

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]) \qquad (*)$$

for any vector fields X, Y. The Lie bracket is defined by

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

Verify this formula by computing each side for $\omega = f\,dg$ where $f, g \in \Omega^0$. Recall $X(fg) = fX(g) + gX(f)$ and $f\,dg(X) = fX(g)$.

Since $(*)$ is independent of a choice of coordinate system, $(*)$ and the similar formula for p-forms show that the definition of d is invariant under change of coordinates.

20. Let i be the inclusion of the unit circle S^1 into the plane R^2. Define a 1-form ω by $\omega = i^*(x\,dy - y\,dx)$. This form ω is called $d\theta$ because, although θ is not a globally defined function on S^1, local choices differ by an additive constant. For example, when $x > 0$, we may set $\theta = \arctan(y/x)$.

(a) Verify that $\omega = d\,\theta$ on the points of S^1 where $x > 0$.

(b) Show any $\alpha \in \Omega^1(S^1)$ can be written as $\alpha = f\omega$ where $f \in \Omega^0(S^1)$.

(c) Show the map $I : \Omega^1(S^1) \longrightarrow R$ is surjective where I is defined by

$$I(f\omega) = \int_{S^1} f\omega = \int_0^{2\pi} f(\theta)\, d\theta.$$

(d) If $\beta = dg = g'(\theta)\, d\theta$ where $g \in \Omega^0(S^1)$, show $I(\beta) = 0$.

(e) Conversely if $I(\alpha) = 0$, let $g(\theta) = \int_0^\theta f(t)\, dt$ and show $\alpha = dg$.

(f) Deduce that $H^1(S^1) = R$. What is $H^p(S^1)$ for other values of p?

21. Let $\phi(x, y) = (ax + by, cx + dy)$ Then $\phi^*(e_1^\# \wedge e_2^\#)$ is a multiple of $e_1^\# \wedge e_2^\#$. Find that multiple.

22. Let $\omega \in \Omega^1(M)$. Assume there exists a nonvanishing function f such that $d(f\omega) = 0$.

(a) If $U \subset M$ is homeomorphic to a ball, find $g \in \Omega^0(U)$ such that $f\omega = dg$.

(b) Using $\omega = (1/f)dg$, show $\omega \wedge d\omega = 0$.

(c) If $\omega = y\, dx - x\, dy + dz$ on R^3, show $\omega \wedge d\omega \neq 0$.

If $\omega = F^\#$ for a nonvanishing vector field F, then $fF = \operatorname{grad} g$. F is orthogonal to the level surfaces of g, f is called an integrating factor. For $x \in U$, $E_x = \{v : v \perp F_x\} \subset T_xM$ is the tangent space at x to a level surface of g. This is the local definition of a codimension one foliation. A theorem of Frobenius says that $\omega \wedge d\omega = 0$ is sufficient for such surfaces to exist.