Outline of axiomatic singular homology theory

Class notes

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A. The axioms of Eilenberg and Steenrod.

There is a sequence of functors from pairs of spaces to abelian groups written $H_q(X, A)$, $H_q(X, \emptyset)$ is written $H_q(X)$, and natural transformations

$$\partial_* : H_{q+1}(X, A) \longrightarrow H_q(A)$$

satisfying:

exactness:

$$\cdots \longrightarrow H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\partial_*} H_{q-1}(A) \longrightarrow \cdots \quad \text{is exact.}$$

homotopy:

$$f \simeq g : (X, A) \longrightarrow (Y, B) \text{ implies } f_* = g_*.$$

excision:

$$\overline{U} \subset A^\circ \text{ implies } H_q(X - U, A - U) \xrightarrow{=} H_q(X, A).$$

dimension:

$$H_q(\text{Point}) = Z \text{ if } q = 0, \text{ and } = 0 \text{ if } q \neq 0.$$

"...the construction of homology groups is a long and diverse story, with a fairly obscure motivation. In contrast, the axioms ... state precisely the ultimate goal... No motivation is offered for the axioms themselves. The beginning student is asked to take these on faith... This should not be difficult, for most of the axioms are quite natural, and their totality possesses sufficient internal beauty to inspire trust in the least credulous."
E & S: *Foundations of Algebraic Topology*, Princeton Univ. Press, 1952, pp *ix, x.*

These axioms were introduced by Eilenberg and Steenrod around 1945 to organize the various homology and cohomology theories which had been developed to extend simplicial homology to more general topological spaces. By 1960 new theories had been found (including K-theory and stable homotopy) which satisfy the (cohomology form of the) axioms except for the dimension axiom. These are called generalized homology theories.

B. Retracts and reduced homology.

An inclusion $i : A \longrightarrow X$ is a retract if there is a map $r : X \longrightarrow A$ with $r \circ i = 1_A$. In that case $r_* \circ i_* = 1_{A*}$ in the short exact sequence

$$0 \longrightarrow H_q(A) \longrightarrow H_q(X) \longrightarrow H_q(X, A) \longrightarrow 0$$

(we say that the sequence splits) and consequently

$$H_q(X) = H_q(A) \oplus H_q(X, A).$$

Let P be a space with one point. Any map $P \longrightarrow X$ is a retract so $H_q(X) \longrightarrow H_q(P)$ is onto. The map $X \longrightarrow P$ is unique. The reduced homology groups are defined by

$$\widetilde{H}_q(X) = \ker\{H_q(X) \longrightarrow H_q(P)\}.$$

$\widetilde{H}_q(X) = H_q(X)$ for $q > 0$. $\widetilde{H}_q(P) = 0$, and $\widetilde{H}_q(D^n) = 0$, for all q, since $D^n \simeq P$.

There is a long exact sequence for reduced homology:

$$\begin{array}{ccccccccc}
\cdots\longrightarrow & \widetilde{H}_q(A) & \longrightarrow & \widetilde{H}_q(X) & \longrightarrow & H_q(X,A) & \longrightarrow & \widetilde{H}_{q-1}(A) & \longrightarrow\cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots\longrightarrow & H_q(A) & \longrightarrow & H_q(X) & \longrightarrow & H_q(X,A) & \longrightarrow & H_{q-1}(A) & \longrightarrow\cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots\longrightarrow & H_q(P) & \longrightarrow & H_q(P) & \longrightarrow & H_q(P,P) & \longrightarrow & H_{q-1}(P) & \longrightarrow\cdots
\end{array}$$

The top row of groups is the kernel of the map from the middle row to the bottom row. The existence of the maps in the top row and exactness follows by diagram chasing.

Taking $A = P$, $\widetilde{H}_q(X) = H_q(X, P)$ and hence $H_0(X) = Z \oplus \widetilde{H}_0(X)$. The reduced homology sequence of the pair (D^n, S^{n-1}) shows $H_q(D^n, S^{n-1}) = \widetilde{H}_{q-1}(S^{n-1})$.

C. Homology of spheres and consequences.

The purse string map $f : (D^n, S^{n-1}) \longrightarrow (S^n, P)$ induces an isomorphism on homology. To see this let

$$U \subset A \subset D^n \quad \text{and} \quad U' \subset A' \subset S^n$$

be defined by $U = \{x : 3/4 < |x| \leq 1\}$, $A = \{x : 1/2 \leq |x| \leq 1\}$, $A' = f(A)$, and $U' = f(U)$. In the diagram

$$\begin{array}{ccc}
(D^n, S^{n-1}) & \xrightarrow{f} & (S^n, P) \\
\downarrow & & \downarrow \\
(D^n, A) & \longrightarrow & (S^n, A') \\
\uparrow & & \uparrow \\
(D^n - U, A - U) & \longrightarrow & (S^n - U', A' - U')
\end{array}$$

the down arrows are homotopy equivalences, the up arrows are excision maps, and the bottom arrow is a homeomorphism. It follows that all maps induce isomorphisms on homology.

Hence the maps $\widetilde{H}_{q-1}(S^{n-1}) \xleftarrow{\partial_*} H_q(D^n, S^{n-1}) \xrightarrow{f_*} H_q(S^n, P) \longleftarrow \widetilde{H}_q(S^n)$ are all isomorphisms. Let $S^0 = \{+1\} \cup \{-1\}$. Then $\widetilde{H}_q(S^0) = H_q(S^0, \{+1\})$ which, by excision is isomorphic to $H_q(\{-1\})$. Thus we have computed

$$\widetilde{H}_q(S^n) = H_q(D^n, S^{n-1}) = \begin{cases} Z, & \text{if } q = n; \\ 0, & \text{otherwise.} \end{cases}$$

Several results of L. E. J. Brouwer are consequences. S^{n-1} is not a retract of D^n. It follows, as for the case of D^2, that any $f : D^n \longrightarrow D^n$ has a fixed point. Spheres and euclidean spaces of different dimensions are not homeomorphic, and in fact no open subset $U \subset R^m$ is homeomorphic to any open subset $V \subset R^n$ if $m \neq n$ (invariance of dimension).

To prove this last result, let P be a one-point subspace of U. Since U is open there is a closed ball $D^m \subset U$ centered at P. Let $W = U - D^m$. Then $\overline{W} = U - (D^m)^\circ \subset U - P$ which is open. Then by excision $H_q(U, U-P) = H_q(D^m, D^m - P) = \widetilde{H}_q(S^{m-1})$ so that the dimension m is determined by the homology groups of the pair $(U, U-P)$. Now if $h : U \longrightarrow V$ is a homeomorphism, take any one-point space P in U and let $Q = h(P)$. The pair $(U, U-P)$ is homeomorphic to $(V, V-P)$ and hence $m = n$.

D. Involutions and degree.

If $X = A \cup B$ is a disjoint union of nonempty open subsets and i_A , i_B are the inclusions, then

$$\begin{array}{ccccccccc} \xrightarrow{0} & H_q(A) & \xrightarrow{i_{A*}} & H_q(X) & \longrightarrow & H_q(X,A) & \xrightarrow{0} \\ & & & & \nwarrow^{i_{B*}} & \uparrow = & \\ & & & & & H_q(B) & \end{array}$$

The vertical map is an isomorphism by excision, so the sequence splits and $H_q(X) = H_q(A) \oplus H_q(B)$.

An involution $t : X \longrightarrow X$ is a map such that $t \circ t = 1_X$. Let $t : S^0 \longrightarrow S^0$ be given by $t(+1) = -1$ and $t(-1) = +1$. Now $H_0(S^0) = H_0(\{-1\}) \oplus H_0(\{+1\})$ and taking the homology of the diagram

$$\begin{array}{ccccc} \{-1\} & \longrightarrow & S^0 & \longleftarrow & \{+1\} \\ \downarrow t & & \downarrow t & & \downarrow t \\ \{+1\} & \longrightarrow & S^0 & \longleftarrow & \{-1\} \end{array}$$

shows that $t_* : H_0(S^0) \longrightarrow H_0(S^0)$ is given by $t_0(a,b) = (b,a)$ where we regard $H_0(S^0) = Z \oplus Z$.

From the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{H}_0(S^0) & \longrightarrow & H_0(S^0) & \longrightarrow & H_0(P) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle t_*} & & \downarrow{\scriptstyle t_*} & & \downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & \widetilde{H}_0(S^0) & \longrightarrow & H_0(S^0) & \longrightarrow & H_0(P) & \longrightarrow & 0 \end{array}$$

we see $t_* = -1 : \widetilde{H}_0(S^0) \longrightarrow \widetilde{H}_0(S^0)$.

The involution t can be extended to spheres and disks. Define $t_1 : R^{n+1} \longrightarrow R^{n+1}$ by $t_1(x_1, x_2, \ldots, x_{n+1}) = (-x_1, x_2, \ldots, x_{n+1})$. Let $S^n = \{x \in R^{n+1} : |x| = 1\}$ and $D^n = \{x \in R^{n+1} : x_{n+1} = 0, |x| \leq \pi\}$. The purse string map can be defined equivariantly; define $f : D^n \longrightarrow S^n$ by

$$f(x) = \begin{cases} \cos|x|\, e_{n+1} + (\sin|x|/|x|)\, x, & \text{if } x \neq 0; \\ e_{n+1}, & \text{if } x = 0. \end{cases}$$

Then $f(\partial D^n) = \{-e_{n+1}\} = P$, t_1 induces an involution of D^n and S^n, and f is equivariant: $f(t_1(x)) = t_1(f(x))$. This gives the diagram:

$$\begin{array}{ccccccccccc} \widetilde{H}_0(S^0) & \xleftarrow{\partial_*} & H_1(D^1, S^0) & \xrightarrow{f_*} & H_1(S^1, P) & \longleftarrow & \widetilde{H}_1(S^1) & \longleftarrow & \cdots & \longleftarrow & \widetilde{H}_n(S^n) \\ \downarrow{\scriptstyle t_{1*}} & & \downarrow{\scriptstyle t_{1*}} & & \downarrow{\scriptstyle t_{1*}} & & \downarrow{\scriptstyle t_{1*}} & & & & \downarrow{\scriptstyle t_{1*}} \\ \widetilde{H}_0(S^0) & \xleftarrow{\partial_*} & H_1(D^1, S^0) & \xrightarrow{f_*} & H_1(S^1, P) & \longleftarrow & \widetilde{H}_1(S^1) & \longleftarrow & \cdots & \longleftarrow & \widetilde{H}_n(S^n) \end{array}$$

where the horizontal maps are all isomorphisms, the groups are all isomorphic to Z, and consequently in each case t_{1*} is multiplication by -1.

Given any map $f : S^n \longrightarrow S^n$ or $f : (D^n, S^{n-1}) \longrightarrow (D^n, S^{n-1})$, we define the degree of f by $f_*(\alpha) = (\deg f)\alpha$ where $\alpha \in \widetilde{H}_n(S^n)$ or $H_n(D^n, S^n)$. The degree has the following properties:

(1) $\deg 1 = 1$

(2) $\deg(f \circ f') = \deg f \, \deg f'$

(3) $f \simeq f'$ implies $\deg f = \deg f'$ (and conversely by a result of H. Hopf)

(4) If f is a homotopy equivalence, then $\deg f = \pm 1$

(5) If $f : (D^n, S^{n-1}) \longrightarrow (D^n, S^{n-1})$, then $\deg f = \deg(f|S^n)$.

Let t_j be the involution of R^{n+1} which changes the sign of the jth coordinate and let s interchange the first and jth coordinates,

$$s(x_1, \ldots, x_{n+1}) = (x_j, x_2, \ldots, x_{j-1}, x_1, x_{j-1}, \ldots, x_n).$$

Then $t_j s = s t_1$, by (4) $\deg s = \pm 1$, and by (2) $\deg t_j = \deg t_1 = -1$ The antipodal map on S^n, $a(x) = -x$, can be written $a = t_1 \circ t_2 \circ \cdots \circ t_{n+1}$ and hence $\deg a = (-1)^{n+1}$.

If $f : S^n \longrightarrow S^n$ has no fixed point, then f is homotopic to a and hence $\deg f = (-1)^{n+1}$, another result of Brouwer. If f takes no point to its antipode, then f is homotopic to 1_{S^n} and $\deg f = 1$. For $f : S^n \longrightarrow S^n$, the Lefschetz number is defined by $L(f) = 1 + (-1)^n \deg f$. If f has no fixed point, we have $L(f) = 0$, so $L(f) \neq 0$ implies f has a fixed point. The Lefschetz fixed point theorem generalizes this fact to a larger class of spaces. For $f : X \longrightarrow X$ the Lefschetz number is defined by

$$L(f) = \sum (-1)^q \text{trace}\,(f_* | H_q(X)).$$

(If φ is an endomorphism of a free abelian group with a basis $e_1, \ldots, e_n$ we can write $\varphi(e_i) = \sum a_i^j e_j$ and define $\text{trace}\,\varphi = \sum a_i^i$.) For example if $X = D^n$, $L(f) = 1$ and f has a fixed point by Brouwer's theorem. On the other hand, if $X = (D^n)^\circ$, $L(f) = 1$, but there are maps f with no fixed point.

A vector field on S^n is a continuous assignment to each point $x \in S^n$ of a vector v_x tangent to S^n at x. We regard v_x as a vector in R^{n+1} which is perpendicular to x. For n odd, and $x = (x_1, \ldots, x_{n+1}) \in S^n$, the assignment $v_x = (-x_2, x_1, \ldots, -x_{n+1}, x_n)$ gives a field of unit tangent vectors. If v_x is a nonvanishing vector field on S^n, then $v_n/|v_n|$ is a unit vector field. Assuming v_x itself is a field of unit vectors, $h(x,t) = \cos(t)\,x + \sin(t)\,v_x$, for $0 \leq t \leq \pi$, is a homotopy from 1_{S^n} to the antipodal map. Hence n must be odd.

On S^3 there are three linearly independent vector fields which give an orthonormal basis for the tangent space at each point:

$$\begin{aligned} v_1(x) &= (-x_2, x_1, -x_4, x_3) \\ v_2(x) &= (-x_3, x_4, x_1, -x_2) \\ v_3(x) &= (-x_4, -x_3, x_2, x_1). \end{aligned}$$

If you regard x as a quaternion of unit length, then these vector fields are ix, jx, and kx.

The tangent bundle of S^n is the space of tangent vectors

$$TS^n = \{(x, v_x) \in S^n \times R^{n+1} : |x| = 1, x \perp v_x\}.$$

The bundle of unit tangent vectors is the Stiefel manifold $V_2(R^{n+1})$. There is a projection map $p : TS^n \longrightarrow S^n$ defined by $p(x, v_x) = x$. For the 3-sphere, the tangent bundle is trivial, that is, it is a product. The map

$$\begin{array}{ccc} S^n \times R^{n+1} & \longrightarrow & TS^n \\ (x, t_1, t_2, t_3) & \mapsto & (x, t_1 v_1 + t_2 v_2 + t_3 v_3) \end{array}$$

is a homeomorphism which commutes with projection onto S^n and is a linear map on each fibre $p^{-1}(x)$. The Cayley numbers give a similar construction for S^7. The spheres of dimensions 1, 3, and 7 are the only ones whose tangent bundles are products by a result of Bott, Milnor, and Kervaire in 1958. The determination of the number of linearly independent vector fields on S^n is due to J. F. Adams in 1962.

E. Additivity of degree.

Let $S^n \vee S^n$ be the one-point union and denote the inclusion maps and retractions by

$$\begin{array}{ccccc} S^n & \xrightarrow{j_1} & S^n \vee S^n & \xleftarrow{j_2} & S^n \\ S^n & \xleftarrow{r_1} & S^n \vee S^n & \xrightarrow{r_2} & S^n. \end{array}$$

From the diagram

$$\begin{array}{ccccccc} \longrightarrow & \widetilde{H}_q(S^n) & \xrightarrow{j_{1*}} & \widetilde{H}_q(S^n \vee S^n) & \longrightarrow & H_q(S^n \vee S^n, S^n) & \longrightarrow \\ & & & \uparrow{\scriptstyle j_{2*}} & & \uparrow{\scriptstyle =} & \\ & & & \widetilde{H}_q(S^n) & \xrightarrow{=} & H_q(S^n, P) & \end{array}$$

where the last vertical arrow is an isomorphism by excision and a homotopy equivalence, we deduce that the sequence splits and

$$\widetilde{H}_q(S^n \vee S^n) = \widetilde{H}_q(S^n) \oplus \widetilde{H}_q(S^n).$$

with the isomorphisms given by $\gamma \mapsto (r_{1*}\gamma, r_{2*}\gamma)$ and $(\alpha, \beta) \mapsto j_{1*}\alpha + j_{2*}\beta$. Hence

$$j_{1*}r_{1*} + j_{2*}r_{2*} = 1_{\widetilde{H}_q(S^n \vee S^n)}.$$

The map $c : S^n \longrightarrow S^n \vee S^n$ collapses the equator to the point which is common to the two spheres. Composing with the retractions above $r_i \circ c \simeq 1_{S^n}$ for $i = 1, 2$. Hence $c_* : \widetilde{H}_q(S^n) \longrightarrow \widetilde{H}_q(S^n \vee S^n)$ satisfies

$$c_*\alpha = (j_{1*}r_{1*} + j_{2*}r_{2*})c_*\alpha = j_{1*}\alpha + j_{2*}\alpha.$$

For $f_i : (S^n, 1) \longrightarrow (X, x_0)$, $i = 1, 2$ define $f_1 * f_2 = (f_1 \vee f_2) \circ c$. Now $(f_1 \vee f_2) \circ j_i = f_i$, $i = 1, 2$ and hence

$$(f_1 * f_2)_*\alpha = (f_1 \vee f_2)_*(j_{1*}\alpha + j_{2*}\alpha) = f_{1*}\alpha + f_{2*}\alpha.$$

In case $X = S^n$ this implies that $\deg(f_1 * f_2) = \deg f_1 + \deg f_2$ Thus beginning with the maps 1_{S^n} and $t_1 : S^n \longrightarrow S^n$ we can construct maps of any degree.

The support of $f : (S^n, 1) \longrightarrow (X, x_0)$ is defined to be $\overline{f^{-1}(X - \{x_0\})}$. The operation $*$ could also be described this way: if the maps f_1 and f_2 are modified by a homotopy so that they are supported on disjoint balls B_1 and $B_2 \subset S^n - \{1\}$, then

$$f_1 * f_2 = \begin{cases} f_1 & \text{on } B_1, \\ f_2 & \text{on } B_2, \\ \varepsilon_{x_0}, & \text{elsewhere.} \end{cases}$$

Let $\pi_n(X, x_0) = [(S^n, 1), (X, x_0)]$ be the set of homotopy classes of maps from S^n to X preserving the base points. The operation $*$ induces a map $\pi_n(X, x_0) \times \pi_n(X, x_0) \longrightarrow \pi_n(X, x_0)$ which in the case $n = 1$ is just the fundamental group structure. For $n > 1$ we will show later that $\pi_n(X, x_0)$ is an abelian group, the nth homotopy group of Hurewicz.

Fix a generator $\gamma \in H_n(S^n)$ and define the Hurewicz map

$$h : \pi_n(X, x_0) \longrightarrow H_n(X)$$

by $h([f]) = f_*(\gamma)$. The formula for $(f_1 * f_2)_* \alpha$ above shows h is a homomorphism. For $X = S^1$ it is an isomorphism. We will show later that for $n = 1$ and X arc-connected, h is onto with kernel equal to the commutator subgroup of $\pi_1(X, x_0)$, so $H_1(X)$ is the abelianization of $\pi_1(X, x_0)$.

As an application we compute the homology of the projective plane, RP^2 which can be constructed by attaching D^2 to S^1 by $p : \partial D^2 \longrightarrow S^1$. There is an identification map $f : (D^2, S^1) \longrightarrow (RP^2, S^1)$ which sends antipodal points on the boundary of D^2 to the same point of RP^2, so $f|\partial D^2 = p$ is the double covering map from S^1 to S^1 which has degree 2. Let $U \subset A \subset D^2$ be as on page two, $f(A) = M$ is a Möbius band, and $U' = f(U)$.

$$\begin{array}{ccc} (D^2, S^1) & \xrightarrow{f} & (RP^2, S^1) \\ \downarrow & & \downarrow \\ (D^2, A) & \longrightarrow & (RP^2, M) \\ \uparrow & & \uparrow \\ (D^2 - U, A - U) & \longrightarrow & (RP^2 - U', M - U'). \end{array}$$

As on page two, the down arrows are homotopy equivalences, the up arrows are excision maps, and the bottom arrow is a homeomorphism, so all maps induce isomorphism in homology. Then we have

$$\begin{array}{ccccccccccc} & & & & Z & & Z & & & & \\ & & & & \| & & \| & & & & \\ 0 & \longrightarrow & 0 & \longrightarrow & H_2(D^2, S^1) & \xrightarrow{=} & H_1(S^1) & \longrightarrow & 0 & & \\ & & & & \downarrow f_* & & \downarrow p_* & & & & \\ 0 & \longrightarrow & H_2(RP^2) & \longrightarrow & H_2(RP^2, S^1) & \xrightarrow{\times 2} & H_1(S^1) & \longrightarrow & H_1(RP^2) & \longrightarrow & 0. \end{array}$$

Since f_* is an isomorphism and p_* is multiplication by two, it follows that the labeled map in the bottom row is multiplication by two. By exactness, $H_2(RP^2) = 0$ and $H_1(RP^2) = Z/2Z$. Extending the diagram to the left, the higher groups $H_n(PR^2) = 0$ for $n \geq 2$. Extending the reduced homology sequence of the pair to the right shows $\widetilde{H}_0(RP^2) = 0$.

F. Exact sequences.

The exact sequence of a triple.

Given a nested triple of spaces $B \subset A \subset X$, the exact sequences of the three pairs (X, A), (X, B), and (A, B) fit into a commutative diagram called a braid with a new sequence called the exact sequence of a triple.

$$
\begin{array}{ccccccccccccc}
H_q(B) & & \longrightarrow & & H_q(X) & & \longrightarrow & & H_q(X,A) & & \longrightarrow & & H_{q-1}(A,B) \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
 & & H_q(A) & & & & H_q(X,B) & & & & H_{q-1}(A) & & \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
H_{q+1}(X,A) & & \longrightarrow & & H_q(A,B) & & \longrightarrow & & H_{q-1}(B) & & \longrightarrow & & H_{q-1}(X)
\end{array}
$$

Each sequence goes diagonally up two steps, right one step, diagonally down two steps, right one step, and so on. The boundary map of the triple, $H_{q+1}(X, A) \longrightarrow H_q(A, B)$, is defined by composition of diagonal arrows. Other maps are either boundary maps in the exact sequence of a pair or are induced by inclusions. To prove exactness of the sequence of a triple first notice that the inclusion $(A, B) \longrightarrow (X, A)$ can be factored in two ways:

$$
\begin{array}{ccccc}
 & & (X,B) & & \\
 & \nearrow & & \searrow & \\
(A,B) & & & & (X,A) \\
 & \searrow & & \nearrow & \\
 & & (A,A) & &
\end{array}
$$

Since $H_q(A, A) = 0$, it follows that the upper composition is zero on homology. The other five steps in the proof are accomplished by diagram chasing.

For an example, the sequence of the triple $X \times P \subset X \times S^{n-1} \subset X \times D^n$ shows that $H_q(X \times D^n, X \times S^{n-1}) \xrightarrow{\cong} H_q(X \times S^n, X \times P)$. Then methods similar to section C show that $H_q(X \times S^n, X \times P) = H_{q-n}(X \times S^0, X \times P)$, which by excision is equal to $H_{q-n}(X)$. It follows that

$$H_q(X \times S^n) = H_q(X) \oplus H_{q-n}(X)$$

which can be used to compute the homology of finite product of spheres.

The Mayer-Vietoris sequence.

If $X = U \cup V$ is the union of two open sets U and V, then $\overline{X - V} = X - V \subset U = U^\circ$ and the excision axiom gives

$$H_q(V, U \cap V) \xrightarrow{\cong} H_q(U \cup V, U).$$

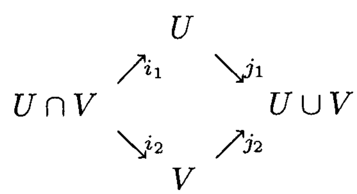
The roles of U and V could be reversed. Whenever the excision maps

$$(V, U \cap V) \longrightarrow (U \cup V, U) \quad \text{and} \quad (U, U \cap V) \longrightarrow (U \cup V, V)$$

induce isomorphisms on homology there is an exact Mayer-Vietoris sequence:

$$\cdots \xrightarrow{\Delta} H_q(U \cap V) \xrightarrow{i} H_q(U) \oplus H_q(V) \xrightarrow{j} H_q(U \cup V) \xrightarrow{\Delta} H_{q-1}(U \cap V) \longrightarrow \cdots$$

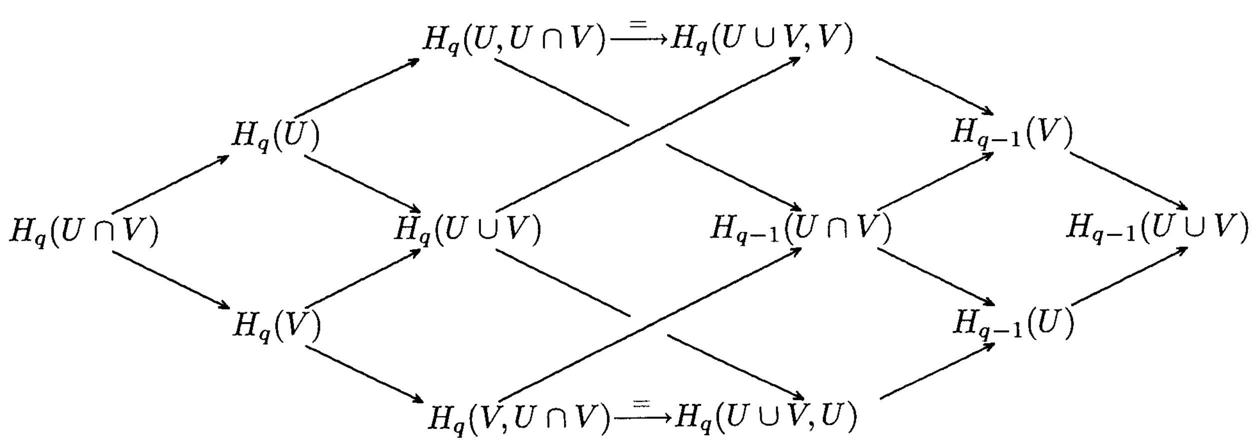
where $i\alpha = (i_{1*}\alpha, -i_{2*}\alpha)$ and $j(\beta, \gamma) = j_{1*}\beta + j_{2*}\gamma$.



The boundary map Δ is defined to be the composition:

$$H_q(U \cup V) \longrightarrow H_q(U \cup V, V) \xleftarrow{\cong} H_q(U, U \cap V) \xrightarrow{\partial_*} H_{q-1}(U \cap V).$$

Reversing the roles of U and V here will change the sign of Δ (see the end of this section). Exactness of the Mayer-Vietoris sequence can be proved with the aid of the diagram:



Another proof using the Barrett-Whitehead lemma is indicated in problem 40. If $U \cap V \neq \emptyset$, the reduced sequences of the pairs lead to the reduced Mayer-Vietoris sequence.

The suspension of a space Y is defined by

$$\Sigma Y = Y \times I \cup P \cup Q / \{Y \times \{0\} = P, Y \times \{1\} = Q\}.$$

where P and Q are one-point spaces. Taking $U = \Sigma Y - Q$ and $V = \Sigma Y - P$ and using the Mayer-Vietoris sequence we find $\widetilde{H}_{q+1}(\Sigma Y) = \widetilde{H}_q(Y)$. Since ΣS^n is homeomorphic to S^{n+1} this gives another way to compute the homology groups of spheres.

For another application we prove a generalization to n-dimensions of the Jordan curve theorem, but with an additional hypothesis. Let U be an open subset of R^n and let $h : S^{n-1} \times (-1, +1) \longrightarrow U$ be a homeomorphism. We call U a bicollar neighborhood of $\Sigma^{n-1} = h(S^{n-1} \times \{0\})$. Σ^{n-1} is a subspace of R^n homeomorphic to an n-sphere in such a way, and this is the additional hypothesis, that it has a neighborhood homeomorphic to $S^{n-1} \times (-1, +1)$. This is true for example if Σ^{n-1} is a differentiably embedded sphere. Taking $V = R^n - \Sigma^{n-1}$, the Mayer-Vietoris sequence shows that $H_0(R^n - \Sigma^{n-1}) = Z \oplus Z$. Now it follows from the axioms that the number of connected components of X is less than or equal to $\operatorname{rank} H_0(X)$, since X contains a discrete set of points as a retract. Singular homology theory has the property that $H_0(X)$ is the free abelian group generated by the path components of X. Thus we have that $R^n - \Sigma^{n-1}$ has two path components. In the case $n = 2$, we have shown that the complement of a simple closed curve in the plane has two path components provided that the curve is bicollared. In the next section we give a proof without this assumption on Σ^{n-1}.

To investigate the definition of Δ insert the group $D = H_q(U \cup V, U \cap V)$ in the center of the diagram above. Then, simplifying notation, we have

$$\begin{array}{ccccccc} & & A & \xrightarrow{=} & B & & \\ & & & \searrow \quad \nearrow & & & \\ C & \xrightarrow{\varphi} & & D & & \xrightarrow{\psi} & E \\ & & & \nearrow \quad \searrow & & & \\ & & F & \xrightarrow{=} & G & & \end{array}$$

where the three straight line compositions are exact. This implies D is the direct sum of the injective images of A and F; write $\delta \in D$ as $\delta = \delta_A + \delta_F$. Then $\Delta(\gamma) = \psi(\varphi(\gamma)_A)$ and, since $\psi(\varphi(\gamma)_A + \varphi(\gamma)_F) = \psi \circ \varphi(\gamma) = 0$, it follows that $\psi(\varphi(\gamma)_F) = -\Delta(\gamma)$.

G. The axiom of compact support.

Singular homology satisfies the axiom of compact support:

(a) Given any $z \in H_q(X, A)$ there is a compact subpair $i_1 : (X_1, A_1) \subset (X, A)$ and an element $z_1 \in H_q(X_1, A_1)$ with $z = i_{1*}z_1$.

(b) Given $z_1 \in H_q(X_1, A_1)$ with $i_{1*}z_1 = 0$, i_1 can be factored through a compact pair $i_1 = j \circ i_2$ where

$$(X_1, A_1) \xrightarrow{j} (X_2, A_2) \xrightarrow{i_2} (X, A)$$

so that $j_* z_1 = 0$.

Axiom (b) follows from axiom (a) and exactness, see Spanier 4.8.12, but both will follow easily from the construction of singular homology. These axioms can be restated in terms of direct limits, but we need only a special case.

Let $\{U_n\}$ be a nested sequence of open subsets of X:

$$U_0 \subset U_1 \subset \cdots \subset U_n \subset \cdots$$

with $X = \bigcup U_n$. Let $j_n : U_n \longrightarrow U_{n+1}$ be the inclusion map.

Consider sequences $\mathbf{x} = \{x_k, x_{k+1}, \ldots\}$ of elements $x_n \in H_q(U_n)$ such that $x_{n+1} = j_n(x_n)$ for $n \geq k$. Two sequences $\mathbf{x}$ and $\mathbf{y}$ are called equivalent if $x_n = y_n$ for all sufficiently large n. Notice that the sequence is determined by its first element and that the sequence determined by x_k is equivalent to the one determined by y_ℓ if and only if there is an integer $m \geq k, \ell$ such that x_k and y_ℓ map to the same element of $H_q(U_m)$ under the maps induced by inclusions. Define the sum $\mathbf{x}+\mathbf{y}$ to be the sequence $\{x_m + y_m, x_{m+1} + y_{m+1}, \ldots\}$ where m is large enough that both x_m and y_m are defined. This makes the set of equivalence classes into an abelian group. The direct limit, $\varinjlim H_q(U_n)$, is defined to be this group.

Let $i_n : U_n \longrightarrow X$ be the inclusion. Then the sequence of maps $\{i_{n*}\}$ gives a map $\varinjlim H_q(U_n) \longrightarrow H_q(X)$. By axiom (a), this map is onto: given an element $z \in H_q(X)$, we have $z_1 \in H_q(X_1)$ for some compact X_1. By compactness $X_1 \subset U_k$ for some k. Let z_1 map to $z_k \in H_q(U_k)$. Then the sequence $\{z_k, z_{k+1}, \ldots\}$ represents an element of $\varinjlim H_q(U_n)$ which maps to $z \in H_q(X)$. Similarly, by axiom (b) this map is onto, hence

$$\varinjlim H_q(U_n) \xrightarrow{=} H_q(X).$$

For a more general description of direct limits see Bredon, Appendix D, or Spanier.

As an application we prove the Jordan-Brouwer separation theorem. This approach is due to Alexander, 1922, cf. Bredon, IV.19.

LEMMA. If $A \subset S^n$ is homeomorphic to a disk D^k, then $\widetilde{H}_q(S^n - A) = 0$ for all q.

The proof is by induction on k. For $k = 0$, $S^n - A$ is homeomorphic to R^n. Suppose the lemma is true for $k-1$, but that $\widetilde{H}_q(S^n - A)$ contains an element $y_0 \neq 0$ for some q. Since $D^k \approx D^{k-1} \times I$, let $h : D^{k-1} \times I \xrightarrow{\approx} A$ and let $A_1 = h(D^{k-1} \times [0, 1/2])$ and $A_1' = h(D^{k-1} \times [1/2, 1])$. Then $A = A_1 \cup A_1'$ and $A_1 \cup A_1'$ is homeomorphic to D^{k-1} so the inductive assumption and the Mayer-Vietoris sequence for the open sets $S^n - A_1$ and $S^1 - A_1'$ give an isomorphism

$$\widetilde{H}_q(S^n - A) \xrightarrow{\equiv} \widetilde{H}_q(S^n - A_1) \oplus \widetilde{H}_q(S^n - A_1').$$

Say $y_0 \mapsto y_1 + y_1'$. Then at least one of y_1 and y_1' is not zero, say $0 \neq y_1 \in \widetilde{H}_q(S^n - A_1)$.

Continuing we get a sequence of nonzero elements $y_0 \mapsto y_1 \mapsto y_2 \mapsto \cdots$ with $y_i \in \widetilde{H}_q(S^n - A_i)$, $A_i = h(D^{k-1} \times I_i)$, and $\bigcap A_i \approx D^{k-1}$.

Since $S^n - \bigcap A_i = \bigcup(S^n - A_i)$ and $S^n - A_i$ is an increasing family of open sets,

$$\varinjlim \widetilde{H}_q(S^n - A_i) = \widetilde{H}_q(S^n - \bigcap A_i)$$

which is zero by induction. But $\{y_0, y_1, \ldots\}$ is a nonzero element in the direct limit. Hence there is no $y_0 \neq 0$.

THEOREM. If $S^m \approx \Sigma^m \subset S^n$, then

$$\widetilde{H}_q(S^n - \Sigma^m) = \begin{cases} Z & \text{for } q = n - m - 1 \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Write S^m as the union of two hemispheres, $S^m = D_1^m \cup D_2^m$ with $D_1^m \cap D_2^m = S^{m-1}$. Let $\Sigma^m = A_1 \cup A_2$ with $A_1 \approx D_1^m$, $A_2 \approx D_2^m$, and let $\Sigma^{m-1} = A_1 \cap A_2$. Using the lemma, the Mayer-Vietoris sequence for $S^n - A_1$, $S^n - A_2$ gives $\widetilde{H}_{q+1}(S^n - \Sigma^{m-1}) = \widetilde{H}_q(S^n - \Sigma^m)$. Applying this result m times gives $\widetilde{H}_q(S^n - \Sigma^m) = \widetilde{H}_{q+m}(S^n - S_0) = \widetilde{H}_{q+m}(S^{n-1})$ and the theorem follows.

In co-dimension one $\widetilde{H}_0(S^n - \Sigma^{n-1}) = Z$, so $S^n - \Sigma^{n-1}$ has two path components, say U and V. Σ^{n-1} is the continuous image of S^{n-1}, hence compact and hence closed in S^n. So $S^n - \Sigma^{n-1}$ is open and locally path connected. Therefore U and V are the connected components of $S^n - \Sigma^{n-1}$ and are open in S^n. This is the Jordan-Brouwer theorem.

The case $\Sigma^1 \subset S^3$ is called a knot; $\widetilde{H}_q(S^3 - \Sigma^1) = Z$ for $q = 1$ and is 0 otherwise.

Further topological remarks:

(1) $\Sigma^{n-1} \subset S^n$ is the common boundary of the two components of its complement, $\Sigma^{n-1} = \partial U = \partial V$.

PROOF. Since U and V are open, $\overline{U} \subset S^n - V$ and $\partial U = \overline{U} - U \subset S^n - (U \cup V) = \Sigma^{n-1}$. Similarly $\partial V \subset \Sigma^{n-1}$. We must show $\Sigma^{n-1} \subset \overline{U} \cap \overline{V}$. Let $x \in \Sigma^{n-1}$ and let N be any open neighborhood of x in S^n. Let $A \subset \Sigma^{n-1} \cap N$ with $\Sigma^{n-1} - A \approx D^{n-1}$. By the lemma $\widetilde{H}_0(S^n - (\Sigma^{n-1} - A)) = 0$, so $S^n - (\Sigma^{n-1} - A)$ is path connected. If $p \in U$ and $q \in V$ there is a path ω in $S^n - (\Sigma^{n-1} - A)$ from p to q which must meet Σ^{n-1} and hence A. Considering the first and last points in which ω meets A we see $A \cap \overline{U} \neq \emptyset$ and $A \cap \overline{V} \neq \emptyset$. Therefore N meets $\overline{U}$ and $\overline{V}$ so $x \in \overline{U} \cap \overline{V}$.

(2) Schoenflies' theorem says that any embedding of S^1 in S^2 extends to a homeomorphism $S^2 \xrightarrow{\approx} S^2$ taking the standard equator of S^2 to the embedded S^1. This is false in higher dimensions; the Alexander horned sphere provides a counterexample.

(3) Invariance of domain (Brouwer).

If U is open in S^n and $f : U \longrightarrow S^n$ is continuous and injective then $f(U)$ is open.

PROOF. Given $y = f(x) \in f(U)$, let A be a closed neighborhood of x, with $A \subset U$ and $A \approx D^n$ so $\partial A \approx S^{n-1}$. Then $S^n - f(A)$ is connected and $S^n - f(\partial A)$ has 2 components. Since f is one-to-one, $f(A) - f(\partial A) = f(A - \partial A)$ which is connected. Then

$$S^n - f(\partial A) = (S^n - f(A)) \cup (f(A) - f(\partial A))$$

exhibits $S^n - f(\partial A)$ as a union of two connected sets, hence these are the components of $S^n - f(\partial A)$. Therefore $f(A - \partial A)$ is open in S^n and is contained in $f(U)$ and hence $f(U)$ is open.

H. CW-spaces and cellular homology.

A CW-space is a Hausdorff space X partitioned into a collection $\{e_\alpha\}$ of disjoint subsets satisfying the following four conditions:

(1) There are maps $F_\alpha : D^{n(\alpha)} \longrightarrow X$ for which $F|(D^{n(\alpha)})^\circ$ is a homeomorphism onto e_α. Define the n-skeleton of X to be $X^{(n)} = \bigcup\{e_\alpha : n(\alpha) \leq n\}$.

(2) $f_\alpha = F_\alpha|S^{n(\alpha)-1}$ maps into $X^{n(\alpha)-1}$.

X is finite if there are finitely many cells e_α. A subset A is a (finite) CW-space if it is a closed set and is a union of (finitely many) e_α's.

(3) (Closure finite) Each point of X is contained in a finite CW-subspace.

(4) (Weak topology) X has the topology of the direct limit of its finite CW-subspaces.

The definition of CW space is due to J. H. C. Whitehead.

Remarks:

(a) (4) says $A \subset X$ is closed if and only if the intersection of A with each finite CW-subspace is closed.

This is equivalent to the condition: $A \cap \overline{e}_\alpha$ is closed for all α.

PROOF. Let K be a finite CW-subspace, say $K = e_1 \cup \ldots \cup e_k$. Then $K = \overline{K} = \overline{e}_1 \cup \ldots \cup \overline{e}_k$ and hence $A \cap K = (A \cap \overline{e}_1) \cup \ldots \cup (A \cap \overline{e}_k)$ is closed.

(b) $F_\alpha(D^{n(\alpha)}) = \bar{e}_\alpha$.

PROOF. $F_\alpha^{-1}(\bar{e}_\alpha) \supset \overline{F_\alpha^{-1}(e_\alpha)} = D^{n(\alpha)}$ so $F_\alpha(D^{n(\alpha)}) \subset \bar{e}_\alpha$. $F_\alpha(D^{n(\alpha)})$ is compact and X is Hausdorff hence $F_\alpha(D^{n(\alpha)})$ is closed so $F_\alpha(D^{n(\alpha)}) \supset \bar{e}_\alpha$.

It follows that if $e_\alpha \subset X^{(n)}$, then $\bar{e}_\alpha \subset X^{(n)}$ so that $X^{(n)}$ is closed and hence a CW-subspace. Also any CW-subspace is itself a CW-space. Further $X = \varinjlim X^{(n)}$.

(c) (3) holds if and only if $\bar{e}_\alpha$ meets only finitely many e_β.

PROOF. A point in e_α lies in a finite CW-subspace K by (3). Since K is a disjoint union of cells, $e_\alpha \subset K$, so $\bar{e}_\alpha \subset K$ and hence $\bar{e}_\alpha$ meets only finitely many cells. Conversely, if $p \in X$ we may assume that p lies in the n-skeleton but not in the $(n-1)$-skeleton. If $n = 0$, $\{p\}$ is a 0-cell which is a finite subspace. If $n > 0$, say p lies in the n-cell e_α. Then $\bar{e}_\alpha - e_\alpha \subset X^{(n-1)}$ by remark (b). Now $\bar{e}_\alpha - e_\alpha$ meets only finitely many cells, $e_1, \ldots, e_k$, by hypothesis and each e_i lies in a finite subspace K_i by induction on n. Hence $e_\alpha \cup K_1 \cup \ldots \cup K_k$ is a finite CW-subspace containing p.

CW-spaces can be constructed by attaching cells. Let Y be a CW-space and $f_\lambda : S^{n-1} \longrightarrow Y^{(n-1)}$ for $\lambda \in \Lambda$. Let $X = Y \cup \bigcup_{f_\lambda} D^n_\lambda$, the quotient space formed from the disjoint union of Y and copies of D^n for each $\lambda \in \Lambda$ by identifying points $x \in \partial D^n_\lambda$ with $f_\lambda(x) \in Y$. Set $e_\lambda = (D^n_\lambda)^\circ$. The map $F_\alpha : D^n \longrightarrow X$ is induced by the identity map $D^n \longrightarrow D^n_\lambda$ and $F_\alpha | S^{n-1} = f_\lambda$.

We must show X is Hausdorff if Y is Hausdorff. The main case in the proof is for points $p, q \in Y$. There are disjoint open sets $U, V \subset Y$ containing p and q. Let

$$U' = U \cup \{x \in e_\lambda : |x| > 0 \text{ and } f_\lambda(x/|x|) \in U\}.$$

Defining V' similarly gives disjoint open sets in X.

PROPOSITION. A compact set $K \subset X$ is contained in a finite CW-subspace.

PROOF. We first show the set $\{\alpha : e_\alpha \cap K \neq \emptyset\}$ is finite. Choose $p_\alpha \in e_\alpha \cap K$ for each e_α meeting K and let $P = \bigcup\{p_\alpha\}$. Then $P \cap \bar{e}_\beta$ is finite by (c) so P is closed by (4). Also any subset of P is closed and hence P is discrete. P is compact, since $P \subset K$, and therefore finite. Now each p_α belongs to a finite subspace by (3) so K is contained in the union of these.

COROLLARY. X compact if and only if X is a finite CW-space, in general $X = \varinjlim K$ over all compact subspaces K of X, and $H_q(X) = \varinjlim H_q(K) = \varinjlim H_q(X^{(n)})$.

PROPOSITION. $\bigoplus_\alpha H_q(D^n, S^{n-1}) \xrightarrow{\Sigma F_\alpha} H_q(X^{(n)}, X^{(n-1)})$ is an isomorphism.

PROOF.

$$
\begin{aligned}
H_q(X^{(n)}, X^{(n-1)}) &\xrightarrow{=} H_q(X^{(n)}, X^{(n)} - P) \quad \text{by homotopy} \\
&\xleftarrow{=} H_q(\bigcup e_\alpha, \bigcup (e_\alpha - p_\alpha)) \quad \text{by excision} \\
&\xleftarrow{=} \bigoplus_\alpha H_q(D^n, S^{n-1}) \quad \text{by homotopy.}
\end{aligned}
$$

We have a space X with a filtration

$$\emptyset = X^{(-1)} \subset X^{(0)} \subset X^{(1)} \subset \ldots \subset X^{(n)} \subset \ldots \subset X$$

such that:

(i) $H_j(X^{(n)}, X^{(n-1)}) = 0$ for $j \neq n$,

(ii) $H_j(X, X^{(q)}) = \varinjlim H_j(X^{(n)}, X^{(q)})$

Set $C_n(X) = H_n(X^{(n)}, X^{(n-1)})$ and define $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$ to be the boundary map of the triple $(X^{(n)}, X^{(n-1)}, X^{(n-2)})$ defined by:

$$
\begin{array}{ccccc}
H_n(X^{(n)}, X^{(n-1)}) & \xrightarrow{\partial} & H_{n-1}(X^{(n-1)}) & \longrightarrow & H_{n-1}(X^{(n-1)}, X^{(n-2)}) \\
\big\downarrow = & & & & \big\downarrow = \\
C_n(X) & & \xrightarrow{\partial_n} & & C_{n-1}(X).
\end{array}
$$

Then $\partial_{n-1}\partial_n = 0$ because

$$H_{n-1}(X^{(n-1)}) \longrightarrow H_{n-1}(X^{(n-1)}, X^{(n-2)}) \longrightarrow H_{n-2}(X^{(n-2)})$$

is exact.

$\{C_n(X), \partial_n\}$ is called the cellular chain complex of X. A map $f : X \longrightarrow Y$ is cellular if $f(X^{(n)}) \subset Y^{(n)}$. Such a map induces a chain map $C_n(X) \longrightarrow C_n(Y)$.

THEOREM. $H_n(C(X)) = H_n(X)$.

PROOF. (1) $H_n(X^{(p)}, X^{(q)}) = 0$ for $p \geq q \geq n$ or $n > p \geq q$ by induction on $p - q$. This is trivial for $p - q = 0$. In the exact sequence of the triple

$$H_n(X^{(q+1)}, X^{(q)}) \longrightarrow H_n(X^{(p)}, X^{(q)}) \longrightarrow H_n(X^{(p)}, X^{(q+1)})$$

the first term is 0 by (i) and the last is 0 by induction.

(2) $H_n(X, X^{(q)}) = 0$ for $q \geq n$ by (1) and (ii).

(3) $H_n(X^{(q)}, X^{(r)}) = H_n(X, X^{(r)})$ for $q > n$ and $q \geq r$ by (2) using the sequence of the triple $(X, X^{(q)}, X^{(r)})$.

(4) Chasing the diagram:

$$\begin{array}{ccccccc}
 & & H_{n+1}(X^{(n+1)},X^{(n)}) & & & & 0 \\
 & & \downarrow \partial & \searrow \partial_{n+1} & & & \downarrow \\
 & 0 \longrightarrow & H_n(X^{(n)}) & \longrightarrow & H_n(X^{(n)},X^{(n-1)}) & \longrightarrow & H_{n-1}(X^{(n-1)}) \\
 & & \downarrow & & & \searrow \partial_n & \downarrow \\
H_n(X) & = & H_n(X^{(n+1)}) & & & & H_{n-1}(X^{(n-1)},X^{(n-2)}) \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}$$

yields $H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$.

The nth group in the cellular chain complex,

$$C_n(X) = H_n(X^{(n)},X^{(n-1)}) \overset{\Sigma F_\alpha}{\longleftarrow} \bigoplus_\alpha H_n(D^n,S^{n-1})$$

can be regarded as the free abelian group generated by the n-dimensional cells of X. We fix a consistent choice of generators using the sequence if isomorphisms from page 4:

$$\widetilde{H}_0(S^0) \longleftarrow H_1(D^1,S^0) \longrightarrow H_1(S^1) \longleftarrow H_2(D^2,S^1) \longrightarrow H_2(S^2) \longleftarrow \cdots$$

taking generators corresponding to $(i_{1*} - i_{-1*})1 \in \widetilde{H}_0(S^0)$.

The space $X^{(n)}/X^{(n-1)} \overset{\approx}{\longrightarrow} \bigvee_\alpha S^n$ is homeomorphic to a bouquet of n-spheres, one for each n-cell in X and the composition

$$(D^n,S^{n-1}) \overset{F_\beta}{\longrightarrow} (X^{(n)},X^{(n-1)}) \longrightarrow \bigvee_\alpha S^n \overset{r_\beta}{\longrightarrow} S^n$$

is the purse string map.

The boundary map ∂_n can be described in terms of the attaching maps f_α. Index the n-cells of X by $\alpha \in A$ and the $(n-1)$-cells by $\beta \in B$. Let $c : X^{(n-1)} \longrightarrow X^{(n-1)}/X^{(n-2)}$ be the collapsing map and $r_\beta : X^{(n-1)}/X^{(n-2)} \longrightarrow S^{n-1}$ retract to the βth cell modulo its boundary. Then the α, β entry in the matrix representing ∂_n is $\deg(r_\beta \circ c \circ f_\alpha)$.

$$\begin{array}{ccccc}
C_n(X) & = & H_n(X^{(n)},X^{(n-1)}) & \overset{=}{\longleftarrow} & \bigoplus_\alpha H_n(D^n,S^{n-1}) \\
 & & \downarrow \partial & & \downarrow = \\
\downarrow \partial_n & & \widetilde{H}_{n-1}(X^{(n-1)}) & \overset{\Sigma f_\alpha}{\longleftarrow} & \bigoplus_\alpha \widetilde{H}_n(S^{n-1}) \\
 & & \downarrow & & \\
C_{n-1}(X) & = & H_{n-1}(X^{(n-1)},X^{(n-2)}) & \overset{=}{\longleftarrow} & \bigoplus_\beta H_{n-1}(D^{n-1},S^{n-2}) \\
 & & \downarrow & & \downarrow = \\
 & & \widetilde{H}_{n-1}(X^{(n-1)}/X^{(n-2)}) & \overset{=}{\longleftarrow} & \bigoplus_\beta \widetilde{H}_{n-1}(S^{n-1})
\end{array}$$

The real projective space RP^n is obtained from RP^{n-1} by attaching one cell of dimension n by the double cover map $f_n : S^{n+1} \longrightarrow RP^{n-1}$. Thus, by induction, RP^n is a CW-space with one cell in each dimension up to n. The n-skeleton of RP^m is RP^n. Simplifying the diagram above we have

$$\begin{array}{ccccccc}
C_n(RP^m) & = & H_n(RP^n, RP^{n-1}) & \xleftarrow{=} & H_n(D^n, S^{n-1}) & & \\
 & & \downarrow \partial & & \downarrow = & & \\
\downarrow \partial_n & & H_{n-1}(RP^{n-1}) & \xleftarrow{f_n} & H_{n-1}(S^{n-1})) & = & Z \\
 & & \downarrow & & & & \\
C_{n-1}(RP^m) & = & H_{n-1}(RP^{n-1}, RP^{n-2}) & \xleftarrow{=} & H_{n-1}(D^{n-1}, S^{n-2}) & = & Z
\end{array}$$

The composition of the maps

$$\begin{array}{ccccccccc}
S^{n-1} & \longrightarrow & RP^{n-1} & \longrightarrow & (RP^{n-1}, RP^{n-2}) & \longleftarrow & (D^{n-1}, S^{n-2}) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & RP^{n-1}/RP^{n-2} & = & D^{n-1}/S^{n-2} & = & S^{n-1}
\end{array}$$

is

$$(1 \vee a)c : S^{n-1} \longrightarrow S^{n-1}$$

which has degree $1 + (-1)^n$.

Thus the groups $C_n = Z$ and the chain complex is

$$\cdots\ Z \xrightarrow{2} Z \xrightarrow{0} Z \xrightarrow{2} Z \xrightarrow{0} Z \longrightarrow 0$$

We find

$$H_j(RP^n) = \begin{cases} Z & \text{for } j = 0, \\ Z/2Z & \text{for } 0 < j < n \text{ and } j \text{ odd}, \\ Z & \text{for } j = n \text{ if } n \text{ is odd}, \\ 0 & \text{otherwise.} \end{cases}$$

Complex projective space $CP^n = CP^{n-1} \cup_h D^{2n}$ attached by the Hopf map. Since $C_q(CP^n) = 0$ for q odd, all ∂_q are zero and

$$H_q(CP^n) = \begin{cases} Z & \text{for } q \text{ even and } 0 \le q \le 2n, \\ 0 & \text{otherwise.} \end{cases}$$

Projective spaces and Hopf maps.

Let $F = R$, C, or the quaternions H. Let $d = \dim_R F$ which is 1, 2, or 4 respectively. F^{n+1} is the vector space of $n+1$-tuples, written $z = (z_0, \ldots, z_n)$, with inner product $z \cdot w = \sum z_j \overline{w}_j$. The projective space FP^n is the space of lines in F^{n+1}; two nonzero vectors z and w lie on the same line if and only if there is a nonzero scaler $\lambda \in F$ such that $w = \lambda z$.

The set of vectors in F^{n+1} of length 1 is homeomorphic to S^{dn+d-1}. The Hopf map $h : S^{dn+d-1} \longrightarrow FP^n$ sends a nonzero vector z to the line $[z]$ containing z. h is onto and $f^{-1}([z]) \approx S^{d-1}$. Let

$$e_q = \{[z] \in FP^n : z_q \neq 0, z_j = 0 \text{ for } j > q\}$$

Then $FP^n = \bigcup_{q=0}^n e_q$.

The unit disk in F^q is homeomorphic to D^{dq}. Define $G : D^{dq} \longrightarrow FP^n$ by

$$G(z_0, \ldots, z_{q-1}) = [z_0, \ldots, z_{q-1}, r, 0, ..., 0] \quad \text{where} \quad r = \sqrt{1 - z \cdot z}.$$

The restriction of G to the interior of the disk is a homeomorphism to the cell e_q and the attaching map $G|\partial D^{dq} : S^{dq-1} \longrightarrow FP^{q-1}$ is the Hopf map, h. The projective line, $FP^1 \approx S^d$. To see this define coordinate charts $f_1 : F \longrightarrow FP^1$ and $f_2 : F \longrightarrow FP^1$ by $f_1(z) = [z, 1]$ and $f_2(z) = [1, \overline{z}]$. Then $f_2^{-1} f_1(z) = \overline{z}^{-1}$ for $z \neq 0$. Regarding the nonzero elements of F as $R^d - \{0\}$ the same coordinate change defines the manifold structure on S^d. Thus there are Hopf maps $S^3 \longrightarrow S^2$ and $S^7 \longrightarrow S^4$.