Outline of axiomatic singular homology theory

Class notes
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A. The axioms of Eilenberg and Steenrod.

There is a sequence of functors from pairs of spaces to abelian groups written \( H_q(X, A) \), \( H_q(X, \emptyset) \) is written \( H_q(X) \), and natural transformations

\[
\partial_* : H_{q+1}(X, A) \to H_q(A)
\]

satisfying:

exactness:

\[
\cdots \to H_q(A) \xrightarrow{i_*} H_q(X) \xrightarrow{j_*} H_q(X, A) \xrightarrow{\partial_*} H_{q-1}(A) \to \cdots \text{ is exact.}
\]

homotopy:

\( f \simeq g : (X, A) \to (Y, B) \) implies \( f_* = g_* \).

excision:

\( U \subseteq A^o \) implies \( H_q(X - U, A - U) \xrightarrow{=} H_q(X, A) \).

dimension:

\( H_q(\text{Point}) = \mathbb{Z} \) if \( q = 0 \), and \( 0 \) if \( q \neq 0 \).

"...the construction of homology groups is a long and diverse story, with a fairly obscure motivation. In contrast, the axioms ... state precisely the ultimate goal... No motivation is offered for the axioms themselves. The beginning student is asked to take these on faith... This should not be difficult, for most of the axioms are quite natural, and their totality possesses sufficient internal beauty to inspire trust in the least credulous."


These axioms were introduced by Eilenberg and Steenrod around 1945 to organize the various homology and cohomology theories which had been developed to extend simplicial homology to more general topological spaces. By 1960 new theories had been found (including K-theory and stable homotopy) which satisfy the (cohomology form of the) axioms except for the dimension axiom. These are called generalized homology theories.
B. Retracts and reduced homology.

An inclusion $i : A \to X$ is a retract if there is a map $r : X \to A$ with $r \circ i = 1_A$. In that case $r_* \circ i_* = 1_{A_*}$ in the short exact sequence

$$0 \to H_q(A) \to H_q(X) \to H_q(X, A) \to 0$$

(we say that the sequence splits) and consequently

$$H_q(X) = H_q(A) \oplus H_q(X, A).$$

Let $P$ be a space with one point. Any map $P \to X$ is a retract so $H_q(X) \to H_q(P)$ is onto. The map $X \to P$ is unique. The reduced homology groups are defined by

$$\tilde{H}_q(X) = \ker\{ H_q(X) \to H_q(P) \}.$$  

$\tilde{H}_q(X) = H_q(X)$ for $q > 0$. $\tilde{H}_q(P) = 0$, and $\tilde{H}_q(D^n) = 0$, for all $q$, since $D^n \simeq P$.

There is a long exact sequence for reduced homology:

$$\cdots \to \tilde{H}_q(A) \to \tilde{H}_q(X) \to H_q(X, A) \to \tilde{H}_{q-1}(A) \to \cdots$$

$$\cdots \to H_q(A) \to H_q(X) \to H_q(X, A) \to H_{q-1}(A) \to \cdots$$

$$\cdots \to H_q(P) \to H_q(X) \to H_q(P, P) \to H_{q-1}(P) \to \cdots$$

The top row of groups is the kernel of the map from the middle row to the bottom row. The existence of the maps in the top row and exactness follows by diagram chasing.

Taking $A = P$, $\tilde{H}_q(X) = H_q(X, P)$ and hence $H_0(X) = \mathbb{Z} \oplus \tilde{H}_0(X)$. The reduced homology sequence of the pair $(D^n, S^{n-1})$ shows $H_q(D^n, S^{n-1}) = \tilde{H}_{q-1}(S^{n-1})$.

C. Homology of spheres and consequences.

The purse string map $f : (D^n, S^{n-1}) \to (S^n, P)$ induces an isomorphism on homology. To see this let

$$U \subset A \subset D^n \text{ and } U' \subset A' \subset S^n$$

be defined by $U = \{ x : 3/4 < |x| \leq 1 \}$, $A = \{ x : 1/2 \leq |x| \leq 1 \}$, $A' = f(A)$, and $U' = f(U)$. In the diagram

$$\begin{align*}
(D^n, S^{n-1}) & \xrightarrow{f} (S^n, P) \\
(D^n, A) & \to (S^n, A') \\
(D^n - U, A - U) & \to (S^n - U', A' - U')
\end{align*}$$
the down arrows are homotopy equivalences, the up arrows are excision maps, and the bottom arrow is a homeomorphism. It follows that all maps induce isomorphisms on homology.

Hence the maps $\tilde{H}_{q-1}(S^{n-1}) \xleftarrow{\partial_*} H_q(D^n, S^{n-1}) \xrightarrow{f_*} H_q(S^n, P) \xleftarrow{\tilde{H}_q} \tilde{H}_q(S^n)$ are all isomorphisms. Let $S^0 = \{+1\} \cup \{-1\}$. Then $\tilde{H}_q(S^0) = H_q(S^0, \{+1\})$ which, by excision is isomorphic to $H_q(\{-1\})$. Thus we have computed

$$\tilde{H}_q(S^n) = H_q(D^n, S^{n-1}) = \begin{cases} Z, & \text{if } q = n; \\ 0, & \text{otherwise}. \end{cases}$$

Several results of L. E. J. Brouwer are consequences. $S^{n-1}$ is not a retract of $D^n$. It follows, as for the case of $D^2$, that any $f : D^n \rightarrow D^n$ has a fixed point. Spheres and euclidean spaces of different dimensions are not homeomorphic, and in fact no open subset $U \subset \mathbb{R}^m$ is homeomorphic to any open subset $V \subset \mathbb{R}^n$ if $m \neq n$ (invariance of dimension).

To prove this last result, let $P$ be a one-point subspace of $U$. Since $U$ is open there is a closed ball $D^m \subset U$ centered at $P$. Let $W = U - D^m$. Then $\tilde{W} = U - (D^m)^0 \subset U - P$ which is open. Then by excision $H_q(U, U - P) = H_q(D^m, D^m - P) = \tilde{H}_q(S^{m-1})$ so that the dimension $m$ is determined by the homology groups of the pair $(U, U - P)$. Now if $h : U \rightarrow V$ is a homeomorphism, take any one-point space $P$ in $U$ and let $Q = h(P)$. The pair $(U, U - P)$ is homeomorphic to $(V, V - P)$ and hence $m = n$.

**D. Involutions and degree.**

If $X = A \cup B$ is a disjoint union of nonempty open subsets and $i_A, i_B$ are the inclusions, then

$$0 \rightarrow H_q(A) \xrightarrow{i_{A*}} H_q(X) \xrightarrow{\partial_*} H_q(X, A) \xrightarrow{0}$$

The vertical map is an isomorphism by excision, so the sequence splits and $H_q(X) = H_q(A) \oplus H_q(B)$.

An involution $t : X \rightarrow X$ is a map such that $t \circ t = 1_X$. Let $t : S^0 \rightarrow S^0$ be given by $t(+1) = -1$ and $t(-1) = +1$. Now $H_0(S^0) = H_0(\{-1\}) \oplus H_0(\{+1\})$ and taking the homology of the diagram

$$\begin{array}{ccc}
\{+1\} & \xrightarrow{t} & \{1\} \\
\downarrow \quad & \quad \downarrow \quad & \quad \downarrow \\
\{+1\} & \xrightarrow{t} & \{-1\}
\end{array}$$

shows that $t_* : H_0(S^0) \rightarrow H_0(S^0)$ is given by $t_0(a, b) = (b, a)$ where we regard $H_0(S^0) = Z \oplus Z$. 

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From the diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & \tilde{H}_0(S^0) & \rightarrow & H_0(S^0) & \rightarrow & H_0(P) \rightarrow 0 \\
\downarrow t_* & & \downarrow t_* & & \downarrow 1 & & \\
0 & \rightarrow & \tilde{H}_0(S^0) & \rightarrow & H_0(S^0) & \rightarrow & H_0(P) \rightarrow 0
\end{array}
\]

we see \( t_* = -1 : \tilde{H}_0(S^0) \rightarrow \tilde{H}_0(S^0) \).

The involution \( t \) can be extended to spheres and disks. Define \( t_1 : R^{n+1} \rightarrow R^{n+1} \) by \( t_1(x_1, x_2, \ldots, x_{n+1}) = (-x_1, x_2, \ldots, x_{n+1}) \). Let \( S^n = \{ x \in R^{n+1} : |x| = 1 \} \) and \( D^n = \{ x \in R^{n+1} : x_{n+1} = 0, |x| \leq \pi \} \). The purse string map can be defined equivariantly; define \( f : D^n \rightarrow S^n \) by

\[
f(x) = \begin{cases} 
\cos |x| e_{n+1} + (\sin |x|/|x|) x, & \text{if } x \neq 0; \\
e_{n+1}, & \text{if } x = 0.
\end{cases}
\]

Then \( f(\partial D^n) = -e_{n+1} = P \), \( t_1 \) induces an involution of \( D^n \) and \( S^n \), and \( f \) is equivariant: \( f(t_1(x)) = t_1(f(x)) \). This gives the diagram:

\[
\begin{array}{cccccc}
\tilde{H}_0(S^0) & \overset{t_*}{\rightarrow} & H_1(D^1, S^0) & \overset{f_*}{\rightarrow} & H_1(S^1, P) & \leftarrow & \tilde{H}_1(S^1) & \leftarrow & \cdots & \leftarrow & \tilde{H}_n(S^n) \\
\downarrow t_* & & \downarrow t_* & & \downarrow t_* & & \downarrow t_* & & \downarrow t_* & & \downarrow t_* \\
\tilde{H}_0(S^0) & \overset{t_*}{\rightarrow} & H_1(D^1, S^0) & \overset{f_*}{\rightarrow} & H_1(S^1, P) & \leftarrow & \tilde{H}_1(S^1) & \leftarrow & \cdots & \leftarrow & \tilde{H}_n(S^n)
\end{array}
\]

where the horizontal maps are all isomorphisms, the groups are all isomorphic to \( Z \), and consequently in each case \( t_* \) is multiplication by \(-1\).

Given any map \( f : S^n \rightarrow S^n \) or \( f : (D^n, S^{n-1}) \rightarrow (D^n, S^{n-1}) \), we define the degree of \( f \) by \( f_* : \tilde{H}_n(S^n) \rightarrow \tilde{H}_n(D^n, S^n) \). The degree has the following properties:

1. \( \deg 1 = 1 \)
2. \( \deg(f \circ f') = \deg f \deg f' \)
3. \( f \approx f' \) implies \( \deg f = \deg f' \) (and conversely by a result of H. Hopf)
4. If \( f \) is a homotopy equivalence, then \( \deg f = \pm 1 \)
5. If \( f : (D^n, S^{n-1}) \rightarrow (D^n, S^{n-1}) \), then \( \deg f = \deg(f|S^n) \).

Let \( t_j \) be the involution of \( R^{n+1} \) which changes the sign of the \( j \)th coordinate and let \( s \) interchange the first and \( j \)th coordinates,

\[
s(x_1, \ldots, x_{n+1}) = (x_j, x_2, \ldots, x_{j-1}, x_1, x_{j-1}, \ldots, x_n).
\]

Then \( t_j s = s t_1 \), by (4) \( \deg s = \pm 1 \), and by (2) \( \deg t_j = \deg t_1 = -1 \). The antipodal map on \( S^n \), \( a(x) = -x \), can be written \( a = t_1 \circ t_2 \circ \cdots \circ t_{n+1} \) and hence \( \deg a = (-1)^{n+1} \).
If \( f : S^n \to S^n \) has no fixed point, then \( f \) is homotopic to a and hence \( \deg f = (-1)^{n+1} \), another result of Brouwer. If \( f \) takes no point to its antipode, then \( f \) is homotopic to \( 1_{S^n} \) and \( \deg f = 1 \). For \( f : S^n \to S^n \), the Lefschetz number is defined by \( L(f) = 1 + (-1)^n \deg f \). If \( f \) has no fixed point, we have \( L(f) = 0 \), so \( L(f) \neq 0 \) implies \( f \) has a fixed point. The Lefschetz fixed point theorem generalizes this fact to a larger class of spaces. For \( f : X \to X \) the Lefschetz number is defined by

\[
L(f) = \sum (-1)^q \text{trace} (f_*|H_q(X)).
\]

(If \( \varphi \) is an endomorphism of a free abelian group with a basis \( e_1, \ldots, e_n \) we can write \( \varphi(e_i) = \sum a_i^j e_j \) and define \( \text{trace} \varphi = \sum a_i^i \).) For example if \( X = D^n \), \( L(f) = 1 \) and \( f \) has a fixed point by Brouwer’s theorem. On the other hand, if \( X = (D^n)^o \), \( L(f) = 1 \), but there are maps \( f \) with no fixed point.

A vector field on \( S^n \) is a continuous assignment to each point \( x \in S^n \) of a vector \( v_x \) tangent to \( S^n \) at \( x \). We regard \( v_x \) as a vector in \( R^{n+1} \) which is perpendicular to \( x \). For \( n \) odd, and \( x = (x_1, \ldots, x_{n+1}) \in S^n \), the assignment \( v_x = (-x_2, x_1, \ldots, -x_{n+1}, x_n) \) gives a field of unit tangent vectors. If \( v_x \) is a nonvanishing vector field on \( S^n \), then \( v_x/|v_x| \) is a unit vector field. Assuming \( v_x \) itself is a field of unit vectors, \( h(x, t) = \cos(t) x + \sin(t) v_x \), for \( 0 \leq t \leq \pi \), is a homotopy from \( 1_{S^n} \) to the antipodal map. Hence \( n \) must be odd.

On \( S^3 \) there are three linearly independent vector fields which give an orthonormal basis for the tangent space at each point:

\[
\begin{align*}
&v_1(x) = (-x_2, x_1, -x_4, x_3) \\
&v_2(x) = (-x_3, x_4, x_1, -x_2) \\
&v_3(x) = (-x_4, -x_3, x_2, x_1).
\end{align*}
\]

If you regard \( x \) as a quaternion of unit length, then these vector fields are \( ix, jx, \) and \( kx \).

The tangent bundle of \( S^n \) is the space of tangent vectors

\[
TS^n = \{(x, v_x) \in S^n \times R^{n+1} : |x| = 1, x \perp v_x \}.
\]

The bundle of unit tangent vectors is the Stiefel manifold \( V_2(R^{n+1}) \). There is a projection map \( p : TS^n \to S^n \) defined by \( p(x, v_x) = x \). For the 3-sphere, the tangent bundle is trivial, that is, it is a product. The map

\[
\begin{array}{ccc}
S^n \times R^{n+1} & \to & TS^n \\
(x, t_1, t_2, t_3) & \mapsto & (x, t_1 v_1 + t_2 v_2 + t_3 v_3)
\end{array}
\]

\[
^3
\]

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is a homeomorphism which commutes with projection onto \( S^n \) and is a linear map on each fibre \( p^{-1}(x) \). The Cayley numbers give a similar construction for \( S^7 \). The spheres of dimensions 1, 3, and 7 are the only ones whose tangent bundles are products by a result of Bott, Milnor, and Kervaire in 1958. The determination of the number of linearly independent vector fields on \( S^n \) is due to J. F. Adams in 1962.

E. Additivity of degree.

Let \( S^n \lor S^n \) be the one-point union and denote the inclusion maps and retraction by

\[
\begin{align*}
S^n & \overset{j_1}{\longrightarrow} S^n \lor S^n \\
S^n & \overset{j_2}{\longrightarrow} S^n
\end{align*}
\]

From the diagram

\[
\begin{align*}
\longrightarrow & \quad \tilde{H}_q(S^n) \overset{j_1\ast}{\longrightarrow} \tilde{H}_q(S^n \lor S^n) \longrightarrow H_q(S^n \lor S^n, S^n) \longrightarrow \\
& \uparrow j_2\ast \\
\tilde{H}_q(S^n) & \overset{=}\longrightarrow H_q(S^n, P)
\end{align*}
\]

where the last vertical arrow is an isomorphism by excision and a homotopy equivalence, we deduce that the sequence splits and

\[
\tilde{H}_q(S^n \lor S^n) = \tilde{H}_q(S^n) \oplus \tilde{H}_q(S^n).
\]

with the isomorphisms given by \( \gamma \mapsto (r_1\ast \gamma, r_2\ast \gamma) \) and \( (\alpha, \beta) \mapsto j_1\ast \alpha + j_2\ast \beta \). Hence

\[
j_1\ast r_1\ast + j_2\ast r_2\ast = 1\tilde{H}_q(S^n \lor S^n).
\]

The map \( c : S^n \to S^n \lor S^n \) collapses the equator to the point which is common to the two spheres. Composing with the retraction above \( r_i \circ c \simeq 1_{S^n} \) for \( i = 1, 2 \). Hence \( c_\ast : \tilde{H}_q(S^n) \to \tilde{H}_q(S^n \lor S^n) \) satisfies

\[
c_\ast \alpha = (j_1\ast r_1\ast + j_2\ast r_2\ast)c_\ast \alpha = j_1\ast \alpha + j_2\ast \alpha.
\]

For \( f_i : (S^n, 1) \to (X, x_0), \ i = 1, 2 \) define \( f_1 \land f_2 = (f_1 \lor f_2) \circ c \). Now \( (f_1 \lor f_2) \circ j_i = f_i, \ i = 1, 2 \) and hence

\[
(f_1 \land f_2)_\ast \alpha = (f_1 \lor f_2)_\ast (j_1\ast \alpha + j_2\ast \alpha) = f_1\ast \alpha + f_2\ast \alpha.
\]

In case \( X = S^n \) this implies that \( \deg(f_1 \land f_2) = \deg f_1 + \deg f_2 \). Thus beginning with the maps \( 1_{S^n} \) and \( t_1 : S^n \to S^n \) we can construct maps of any degree.
The support of \( f : (S^n, 1) \to (X, x_0) \) is defined to be \( \overline{f^{-1}(X - \{x_0\})} \). The operation \(*\) could also be described this way: if the maps \( f_1 \) and \( f_2 \) are modified by a homotopy so that they are supported on disjoint balls \( B_1 \) and \( B_2 \subset S^n - \{1\} \), then

\[
f_1 \ast f_2 = \begin{cases} 
  f_1 & \text{on } B_1, \\
  f_2 & \text{on } B_2, \\
  \varepsilon_{x_0}, & \text{elsewhere.}
\end{cases}
\]

Let \( \pi_n(X, x_0) = [(S^n, 1), (X, x_0)] \) be the set of homotopy classes of maps from \( S^n \) to \( X \) preserving the base points. The operation \(*\) induces a map \( \pi_n(X, x_0) \times \pi_n(X, x_0) \to \pi_n(X, x_0) \) which in the case \( n = 1 \) is just the fundamental group structure. For \( n > 1 \) we will show later that \( \pi_n(X, x_0) \) is an abelian group, the \( n \)th homotopy group of Hurewicz.

Fix a generator \( \gamma \in H_n(S^n) \) and define the Hurewicz map

\[
h : \pi_n(X, x_0) \to H_n(X)
\]

by \( h([f]) = f_\ast(\gamma) \). The formula for \((f_1 \ast f_2)_\ast \alpha\) above shows \( h \) is a homomorphism. For \( X = S^1 \) it is an isomorphism. We will show later that for \( n = 1 \) and \( X \) arc-connected, \( h \) is onto with kernel equal to the commutator subgroup of \( \pi_1(X, x_0) \), so \( H_1(X) \) is the abelianization of \( \pi_1(X, x_0) \).

As an application we compute the homology of the projective plane, \( RP^2 \) which can be constructed by attaching \( D^2 \) to \( S^1 \) by \( p : \partial D^2 \to S^1 \). There is an identification map \( f : (D^2, S^1) \to (RP^2, S^1) \) which sends antipodal points on the boundary of \( D^2 \) to the same point of \( RP^2 \), so \( f|\partial D^2 = p \) is the double covering map from \( S^1 \) to \( S^1 \) which has degree 2. Let \( U \subset A \subset D^2 \) be as on page two, \( f(A) = M \) is a Möbius band, and \( U' = f(U) \).

\[
\begin{array}{ccc}
(D^2, S^1) & \xrightarrow{f} & (RP^2, S^1) \\
\downarrow & & \downarrow \\
(D^2, A) & \longrightarrow & (RP^2, M) \\
\uparrow & & \uparrow \\
(D^2 - U, A - U) & \to & (RP^2 - U', M - U').
\end{array}
\]

As on page two, the down arrows are homotopy equivalences, the up arrows are excision maps, and the bottom arrow is a homeomorphism, so all maps induce isomorphism in homology. Then we have

\[
\begin{array}{cccc}
0 & \to & 0 & \to H_2(D^2, S^1) \xrightarrow{\cong} H_1(S^1) & \to 0 \\
\downarrow f_* \quad & & & \downarrow p_* \\
0 & \to H_2(RP^2) & \to H_2(RP^2, S^1) \xrightarrow{\times^2} H_1(S^1) & \to H_1(RP^2) & \to 0.
\end{array}
\]
Since $f_*$ is an isomorphism and $p_*$ is multiplication by two, it follows that the labeled map in the bottom row is multiplication by two. By exactness, $H_2(RP^2) = 0$ and $H_1(RP^2) = \mathbb{Z}/2\mathbb{Z}$. Extending the diagram to the left, the higher groups $H_n(PR^2) = 0$ for $n \geq 2$. Extending the reduced homology sequence of the pair to the right shows $\tilde{H}_0(RP^2) = 0$.

F. Exact sequences.

The exact sequence of a triple.

Given a nested triple of spaces $B \subset A \subset X$, the exact sequences of the three pairs $(X, A), (X, B),$ and $(A, B)$ fit into a commutative diagram called a braid with a new sequence called the exact sequence of a triple.

$$
\begin{array}{cccccc}
H_q(B) & \longrightarrow & H_q(X) & \longrightarrow & H_q(X, A) & \longrightarrow & H_{q-1}(A, B) \\
\downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \\
H_q(A) & \quad & H_q(X, B) & \quad & H_{q-1}(A) & \quad & \\
\uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \\
H_{q+1}(X, A) & \longrightarrow & H_{q}(A, B) & \longrightarrow & H_{q-1}(B) & \longrightarrow & H_{q-1}(X) \\
\end{array}
$$

Each sequence goes diagonally up two steps, right one step, diagonally down two steps, right one step, and so on. The boundary map of the triple, $H_{q+1}(X, A) \rightarrow H_q(A, B)$, is defined by composition of diagonal arrows. Other maps are either boundary maps in the exact sequence of a pair or are induced by inclusions. To prove exactness of the sequence of a triple first notice that the inclusion $(A, B) \rightarrow (X, A)$ can be factored in two ways:

$$
\begin{array}{c}
\begin{array}{c}
(X, B) \\
\end{array} \\
\end{array}
\begin{array}{c}
\begin{array}{c}
(A, B) \quad (X, A) \\
\end{array} \\
\end{array}
\begin{array}{c}
\begin{array}{c}
(A, A) \\
\end{array} \\
\end{array}
$$

Since $H_q(A, A) = 0$, it follows that the upper composition is zero on homology. The other five steps in the proof are accomplished by diagram chasing.

For an example, the sequence of the triple $X \times P \subset X \times S^{n-1} \subset X \times D^n$ shows that $H_q(X \times D^n, X \times S^{n-1}) \rightarrow H_q(X \times S^{n}, X \times P)$. Then methods similar to section C show that $H_q(X \times S^{n}, X \times P) = H_{q-n}(X \times S^{n}, X \times P)$, which by excision is equal to $H_{q-n}(X)$. It follows that

$$H_q(X \times S^n) = H_q(X) \oplus H_{q-n}(X)$$

which can be used to compute the homology of finite product of spheres.
The Mayer-Vietoris sequence.

If $X = U \cup V$ is the union of two open sets $U$ and $V$, then $X - V = X - V \subset U = U^\circ$ and the excision axiom gives

$$H_q(V, U \cap V) \xrightarrow{=} H_q(U \cup V, U).$$

The roles of $U$ and $V$ could be reversed. Whenever the excision maps

$$(V, U \cap V) \rightarrow (U \cup V, U) \quad \text{and} \quad (U, U \cap V) \rightarrow (U \cup V, V)$$

induce isomorphisms on homology there is an exact Mayer-Vietoris sequence:

$$\ldots \xrightarrow{\Delta} H_q(U \cap V) \xrightarrow{i} H_q(U) \oplus H_q(V) \xrightarrow{j} H_q(U \cup V) \xrightarrow{\Delta} H_{q-1}(U \cap V) \rightarrow \ldots$$

where $i\alpha = (i_1\alpha, -i_2\alpha)$ and $j(\beta, \gamma) = j_1\beta + j_2\gamma$.

The boundary map $\Delta$ is defined to be the composition:

$$H_q(U \cup V) \rightarrow H_q(U \cup V, V) \xrightarrow{=} H_q(U, U \cap V) \xrightarrow{\beta} H_{q-1}(U \cap V).$$

Reversing the roles of $U$ and $V$ here will change the sign of $\Delta$ (see the end of this section). Exactness of the Mayer-Vietoris sequence can be proved with the aid of the diagram:

Another proof using the Barrett-Whitehead lemma is indicated in problem 40. If $U \cap V \neq \emptyset$, the reduced sequences of the pairs lead to the reduced Mayer-Vietoris sequence.
The suspension of a space $Y$ is defined by

$$
\Sigma Y = Y \times I \cup P \cup Q/\{Y \times \{0\} = P, Y \times \{1\} = Q\}.
$$

where $P$ and $Q$ are one-point spaces. Taking $U = \Sigma Y - Q$ and $V = \Sigma Y - P$ and using the Mayer-Vietoris sequence we find $\tilde{H}_{q+1}(\Sigma Y) = \tilde{H}_q(Y)$. Since $\Sigma S^n$ is homeomorphic to $S^{n+1}$ this gives another way to compute the homology groups of spheres.

For another application we prove a generalization to $n$-dimensions of the Jordan curve theorem, but with an additional hypothesis. Let $U$ be an open subset of $R^n$ and let $h : S^{n-1} \times (-1, +1) \rightarrow U$ be a homeomorphism. We call $U$ a bicollar neighborhood of $\Sigma^{n-1} = h(S^{n-1} \times \{0\})$. $\Sigma^{n-1}$ is a subspace of $R^n$ homeomorphic to an $n$-sphere in such a way, and this is the additional hypothesis, that it has a neighborhood homeomorphic to $S^{n-1} \times (-1, +1)$. This is true for example if $\Sigma^{n-1}$ is a differentiably embedded sphere. Taking $V = R^n - \Sigma^{n-1}$, the Mayer-Vietoris sequence shows that $H_0(R^n - \Sigma^{n-1}) = Z \oplus Z$. Now it follows from the axioms that the number of connected components of $X$ is less than or equal to rank $H_0(X)$, since $X$ contains a discrete set of points as a retract. Singular homology theory has the property that $H_0(X)$ is the free abelian group generated by the path components of $X$. Thus we have that $R^n - \Sigma^{n-1}$ has two path components. In the case $n = 2$, we have shown that the complement of a simple closed curve in the plane has two path components provided that the curve is bicollared. In the next section we give a proof without this assumption on $\Sigma^{n-1}$.

To investigate the definition of $\Delta$ insert the group $D = H_q(U \cup V, U \cap V)$ in the center of the diagram above. Then, simplifying notation, we have

$$
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow \varphi & & \downarrow \psi \\
C & \rightarrow & E \\
\downarrow \varphi & & \downarrow \psi \\
F & \rightarrow & G
\end{array}
$$

where the three straight line compositions are exact. This implies $D$ is the direct sum of the injective images of $A$ and $F$; write $\delta \in D$ as $\delta = \delta_A + \delta_F$. Then $\Delta(\gamma) = \psi(\varphi(\gamma)_A)$ and, since $\psi(\varphi(\gamma)_A + \varphi(\gamma)_F) = \psi \circ \varphi(\gamma) = 0$, it follows that $\psi(\varphi(\gamma)_F) = -\Delta(\gamma)$. 
G. The axiom of compact support.

Singular homology satisfies the axiom of compact support:

(a) Given any \( z \in H_q(X, A) \) there is a compact subpair \( i_1 : (X_1, A_1) \subset (X, A) \) and an element \( z_1 \in H_q(X_1, A_1) \) with \( z = i_{1*}z_1 \).

(b) Given \( z_1 \in H_q(X_1, A_1) \) with \( i_{1*}z_1 = 0 \), \( i_1 \) can be factored through a compact pair \( i_1 = j \circ i_2 \) where

\[
(X_1, A_1) \xrightarrow{j} (X_2, A_2) \xrightarrow{i_2} (X, A)
\]

so that \( j_{*}z_1 = 0 \).

Axiom (b) follows from axiom (a) and exactness, see Spanier 4.8.12, but both will follow easily from the construction of singular homology. These axioms can be restated in terms of direct limits, but we need only a special case.

Let \( \{U_n\} \) be a nested sequence of open subsets of \( X \):

\[
U_0 \subset U_1 \subset \cdots \subset U_n \subset \cdots
\]

with \( X = \bigcup U_n \). Let \( j_n : U_n \to U_{n+1} \) be the inclusion map.

Consider sequences \( x = \{x_k, x_{k+1}, \ldots\} \) of elements \( x_n \in H_q(U_n) \) such that \( x_{n+1} = j_{n*}(x_n) \) for \( n \geq k \). Two sequences \( x \) and \( y \) are called equivalent if \( x_n = y_n \) for all sufficiently large \( n \). Notice that the sequence is determined by its first element and that the sequence determined by \( x_k \) is equivalent to the one determined by \( y_{\ell} \) if and only if there is an integer \( m \geq k, \ell \) such that \( x_k \) and \( y_{\ell} \) map to the same element of \( H_q(U_m) \) under the maps induced by inclusions. Define the sum \( x + y \) to be the sequence \( \{x_m + y_m, x_{m+1} + y_{m+1}, \ldots\} \) where \( m \) is large enough that both \( x_m \) and \( y_m \) are defined. This makes the set of equivalence classes into an abelian group. The direct limit, \( \lim \overrightarrow{H_q(U_n)} \), is defined to be this group.

Let \( i_n : U_n \to X \) be the inclusion. Then the sequence of maps \( \{i_n\} \) gives a map \( \lim \overrightarrow{H_q(U_n)} \to H_q(X) \). By axiom (a), this map is onto: given an element \( z \in H_q(X) \), we have \( z_1 \in H_q(X_1) \) for some compact \( X_1 \). By compactness \( X_1 \subset U_k \) for some \( k \). Let \( z_1 \) map to \( z_k \in H_q(U_k) \). Then the sequence \( \{z_k, z_{k+1}, \ldots\} \) represents an element of \( \lim \overrightarrow{H_q(U_n)} \) which maps to \( z \in H_q(X) \). Similarly, by axiom (b) this map is onto, hence

\[
\lim \overrightarrow{H_q(U_n)} \overset{\sim}{\longrightarrow} H_q(X).
\]

For a more general description of direct limits see Bredon, Appendix D, or Spanier.

As an application we prove the Jordan-Brouwer separation theorem. This approach is due to Alexander, 1922, cf. Bredon, IV.19.

**Lemma.** If \( A \subset S^n \) is homeomorphic to a disk \( D^k \), then \( H_q(S^n - A) = 0 \) for all \( q \).
The proof is by induction on \( k \). For \( k = 0 \), \( S^n - A \) is homeomorphic to \( R^n \). Suppose the lemma is true for \( k - 1 \), but that \( \tilde{H}_q(S^n - A) \) contains an element \( y_0 \neq 0 \) for some \( q \). Since \( D^k \approx D^{k-1} \times I \), let \( h : D^{k-1} \times I \xrightarrow{\sim} A \) and let \( A_1 = h(D^{k-1} \times [0,1/2]) \) and \( A'_1 = h(D^{k-1} \times [1/2,1]) \). Then \( A = A_1 \cup A'_1 \) and \( A_1 \cup A'_1 \) is homeomorphic to \( D^{k-1} \) so the inductive assumption and the Mayer-Vietoris sequence for the open sets \( S^n - A_1 \) and \( S^1 - A'_1 \) give an isomorphism

\[
\tilde{H}_q(S^n - A) \xrightarrow{\approx} \tilde{H}_q(S^n - A_1) \oplus \tilde{H}_q(S^n - A'_1).
\]

Say \( y_0 \rightarrow y_1 + y'_1 \). Then at least one of \( y_1 \) and \( y'_1 \) is not zero, say \( 0 \neq y_1 \in \tilde{H}_q(S^n - A_1) \).

Continuing we get a sequence of nonzero elements \( y_0 \rightarrow y_1 \rightarrow y_2 \rightarrow \cdots \) with \( y_i \in \tilde{H}_q(S^n - A_i) \), \( A_i = h(D^{k-1} \times I_i) \), and \( \bigcap A_i \approx D^{k-1} \).

Since \( S^n - \bigcap A_i = \bigcup (S^n - A_i) \) and \( S^n - A_i \) is an increasing family of open sets,

\[
\lim_{\rightarrow} \tilde{H}_q(S^n - A_i) = \tilde{H}_q(S^n - \bigcap A_i)
\]

which is zero by induction. But \( \{y_0, y_1, \ldots\} \) is a nonzero element in the direct limit. Hence there is no \( y_0 \neq 0 \).

**Theorem.** If \( S^m \approx \Sigma^m \subset S^n \), then

\[
\tilde{H}_q(S^n - \Sigma^m) = \begin{cases} Z & \text{for } q = n - m - 1 \\ 0 & \text{otherwise.} \end{cases}
\]

**Proof.** Write \( S^m \) as the union of two hemispheres, \( S^m = D_1^m \cup D_2^m \) with \( D_1^m \cap D_2^m = S^{m-1} \). Let \( \Sigma^m = A_1 \cup A_2 \) with \( A_1 \approx D_1^m \), \( A_2 \approx D_2^m \), and let \( \Sigma^{m-1} = A_1 \cap A_2 \). Using the lemma, the Mayer-Vietoris sequence for \( S^n - A_1, S^n - A_2 \) gives \( \tilde{H}_{q+1}(S^n - \Sigma^{m-1}) = \tilde{H}_q(S^n - \Sigma^m) \). Applying this result \( m \) times gives \( \tilde{H}_q(S^n - \Sigma^m) = \tilde{H}_{q+m}(S^n - S_0) = \tilde{H}_{q+m}(S^{n-1}) \) and the theorem follows.

In co-dimension one \( \tilde{H}_0(S^n - \Sigma^{n-1}) = Z \), so \( S^n - \Sigma^{n-1} \) has two path components, say \( U \) and \( V \). \( \Sigma^{n-1} \) is the continuous image of \( S^{n-1} \), hence compact and hence closed in \( S^n \). So \( S^n - \Sigma^{n-1} \) is open and locally path connected. Therefore \( U \) and \( V \) are the connected components of \( S^n - \Sigma^{n-1} \) and are open in \( S^n \). This is the Jordan-Brouwer theorem.

The case \( \Sigma^1 \subset S^3 \) is called a knot; \( \tilde{H}_q(S^3 - \Sigma^1) = Z \) for \( q = 1 \) and is 0 otherwise.

Further topological remarks:

1. \( \Sigma^{n-1} \subset S^n \) is the common boundary of the two components of its complement, \( \Sigma^{n-1} = \partial U = \partial V \).

**Proof.** Since \( U \) and \( V \) are open, \( \overline{U} \subset S^n - V \) and \( \partial U = \overline{U} - U \subset S^n - (U \cup V) = \Sigma^{n-1} \). Similarly \( \partial V \subset \Sigma^{n-1} \). We must show \( \Sigma^{n-1} \subset \overline{U} \cap \overline{V} \). Let \( x \in \Sigma^{n-1} \) and let \( N \) be any
open neighborhood of $x$ in $S^n$. Let $A \subset \Sigma^{n-1} \cap N$ with $\Sigma^{n-1} - A \approx D^{n-1}$. By the lemma $\tilde{H}_0(S^n - (\Sigma^{n-1} - A)) = 0$, so $S^n - (\Sigma^{n-1} - A)$ is path connected. If $p \in U$ and $q \in V$ there is a path $\omega$ in $S^n - (\Sigma^{n-1} - A)$ from $p$ to $q$ which must meet $\Sigma^{n-1}$ and hence $A$. Considering the first and last points in which $\omega$ meets $A$ we see $A \cap \overline{U} \neq \emptyset$ and $A \cap \overline{V} \neq \emptyset$. Therefore $N$ meets $\overline{U}$ and $\overline{V}$ so $x \in \overline{U} \cap \overline{V}$.

(2) Schoenflies' theorem says that any embedding of $S^1$ in $S^2$ extends to a homeomorphism $S^2 \cong S^2$ taking the standard equator of $S^2$ to the embedded $S^1$. This is false in higher dimensions; the Alexander horned sphere provides a counterexample.

(3) Invariance of domain (Brouwer).

If $U$ is open in $S^n$ and $f : U \to S^n$ is continuous and injective then $f(U)$ is open.

**Proof.** Given $y = f(x) \in f(U)$, let $A$ be a closed neighborhood of $x$, with $A \subset U$ and $A \approx D^n$ so $\partial A \approx S^{n-1}$. Then $S^n - f(A)$ is connected and $S^n - f(\partial A)$ has 2 components. Since $f$ is one-to-one, $f(A) - f(\partial A) = f(A - \partial A)$ which is connected. Then

$$S^n - f(\partial A) = (S^n - f(A)) \cup (f(A) - f(\partial A))$$

exhibits $S^n - f(\partial A)$ as a union of two connected sets, hence these are the components of $S^n - f(\partial A)$. Therefore $f(A - \partial A)$ is open in $S^n$ and is contained in $f(U)$ and hence $f(U)$ is open.

**H. CW-spaces and cellular homology.**

A CW-space is a Hausdorff space $X$ partitioned into a collection $\{e_\alpha\}$ of disjoint subsets satisfying the following four conditions:

1. There are maps $F_\alpha : D^n(\alpha) \to X$ for which $F_\alpha|D^n(\alpha)^c$ is a homeomorphism onto $e_\alpha$.
   Define the $n$-skeleton of $X$ to be $X^{(n)} = \cup \{e_\alpha : n(\alpha) \leq n\}$.

2. $f_\alpha = F_\alpha|S^{n(\alpha)-1}$ maps into $X^{n(\alpha)-1}$.
   $X$ is finite if there are finitely many cells $e_\alpha$. A subset $A$ is a (finite) CW-space if it is a closed set and is a union of (finitely many) $e_\alpha$'s.

3. (Closure finite) Each point of $X$ is contained in a finite CW-subspace.

4. (Weak topology) $X$ has the topology of the direct limit of its finite CW-subspaces.

The definition of CW space is due to J. H. C. Whitehead.

Remarks:

(a) (4) says $A \subset X$ is closed if and only if the intersection of $A$ with each finite CW-subspace is closed.

This is equivalent to the condition: $A \cap \bar{e}_\alpha$ is closed for all $\alpha$.

**Proof.** Let $K$ be a finite CW-subspace, say $K = e_1 \cup \ldots \cup e_k$. Then $K = \overline{K} = \bar{e}_1 \cup \ldots \cup \bar{e}_k$ and hence $A \cap K = (A \cap \bar{e}_1) \cup \ldots \cup (A \cap \bar{e}_k)$ is closed.

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(b) \( F_\alpha(D^n(\alpha)) = \overline{e}_\alpha \).

**Proof.** \( F_\alpha^{-1}(\overline{e}_\alpha) \supset \overline{F_\alpha^{-1}(e_\alpha)} = D^n(\alpha) \) so \( F_\alpha(D^n(\alpha)) \subset \overline{e}_\alpha \). \( F_\alpha(D^n(\alpha)) \) is compact and \( X \) is Hausdorff hence \( F_\alpha(D^n(\alpha)) \) is closed so \( F_\alpha(D^n(\alpha)) \supset \overline{e}_\alpha \).

It follows that if \( e_\alpha \subset X^{(n)} \), then \( \overline{e}_\alpha \subset X^{(n)} \) so that \( X^{(n)} \) is closed and hence a CW-subspace. Also any CW-subspace is itself a CW-space. Further \( X = \lim_{\longrightarrow} X^{(n)} \).

(c) (3) holds if and only if \( \overline{e}_\alpha \) meets only finitely many \( e_\beta \).

**Proof.** A point in \( e_\alpha \) lies in a finite CW-subspace \( K \) by (3). Since \( K \) is a disjoint union of cells, \( e_\alpha \subset K \), so \( \overline{e}_\alpha \subset K \) and hence \( \overline{e}_\alpha \) meets only finitely many cells. Conversely, if \( p \in X \) we may assume that \( p \) lies in the \( n \)-skeleton but not in the \( (n-1) \)-skeleton. If \( n = 0 \), \( \{p\} \) is a 0-cell which is a finite subspace. If \( n > 0 \), say \( p \) lies in the \( n \)-cell \( e_\alpha \). Then \( \overline{e}_\alpha - e_\alpha \subset X^{(n-1)} \) by remark (b). Now \( \overline{e}_\alpha - e_\alpha \) meets only finitely many cells, \( e_1, \ldots, e_k \), by hypothesis and each \( e_i \) lies in a finite subspace \( K_i \) by induction on \( n \). Hence \( e_\alpha \cup K_1 \cup \ldots \cup K_k \) is a finite CW-subspace containing \( p \).

CW-spaces can be constructed by attaching cells. Let \( Y \) be a CW-space and \( f_\lambda : S^{n-1} \rightarrow Y^{(n-1)} \) for \( \lambda \in \Lambda \). Let \( X = Y \cup \bigcup_{\lambda} D^n_\lambda \), the quotient space formed from the disjoint union of \( Y \) and copies of \( D^n \) for each \( \lambda \in \Lambda \) by identifying points \( x \in \partial D^n_\lambda \) with \( f_\lambda(x) \in Y \). Set \( e_\lambda = (D^n_\lambda)^\circ \). The map \( F_\alpha : D^n \rightarrow X \) is induced by the identity map \( D^n \rightarrow D^n_\lambda \) and \( F_\alpha|S^{n-1} = f_\lambda \).

We must show \( X \) is Hausdorff if \( Y \) is Hausdorff. The main case in the proof is for points \( p, q \in Y \). There are disjoint open sets \( U, V \subset Y \) containing \( p \) and \( q \). Let

\[ U' = U \cup \{x \in e_\lambda : |x| > 0 \text{ and } f_\lambda(x/|x|) \in U\}. \]

Defining \( V' \) similarly gives disjoint open sets in \( X \).

**Proposition.** A compact set \( K \subset X \) is contained in a finite CW-subspace.

**Proof.** We first show the set \( \{\alpha : e_\alpha \cap K \neq \emptyset\} \) is finite. Choose \( p_\alpha \in e_\alpha \cap K \) for each \( e_\alpha \) meeting \( K \) and let \( P = \bigcup\{p_\alpha\} \). Then \( P \cap \overline{e}_\beta \) is finite by (c) so \( P \) is closed by (4). Also any subset of \( P \) is closed and hence \( P \) is discrete. \( P \) is compact, since \( P \subset K \), and therefore finite. Now each \( p_\alpha \) belongs to a finite subspace by (3) so \( K \) is contained in the union of these.

**Corollary.** \( X \) compact if and only if \( X \) is a finite CW-space, in general \( X = \lim_{\longrightarrow} K \) over all compact subspaces \( K \) of \( X \), and \( H_q(X) = \lim_{\longrightarrow} H_q(K) = \lim_{\longrightarrow} H_q(X^{(n)}) \).

**Proposition.** \( \bigoplus_{\alpha} H_q(D^n, S^{n-1}) \xrightarrow{\Sigma F_\alpha} H_q(X^{(n)}, X^{(n-1)}) \) is an isomorphism.
Proof.

\[ H_q(X^{(n)}, X^{(n-1)}) \cong H_q(X^{(n)}, X^{(n)} - P) \quad \text{by homotopy} \]
\[ \cong H_q(\bigcup e_\alpha, \bigcup (e_\alpha - p_\alpha)) \quad \text{by excision} \]
\[ \cong \bigoplus_\alpha H_q(D^n, S^{n-1}) \quad \text{by homotopy.} \]

We have a space \( X \) with a filtration

\[ \emptyset = X^{(-1)} \subset X^{(0)} \subset X^{(1)} \subset \ldots \subset X^{(n)} \subset \ldots \subset X \]

such that:

(i) \( H_j(X^{(n)}, X^{(n-1)}) = 0 \) for \( j \neq n \),

(ii) \( H_j(X, X^{(q)}) = \lim_{\rightarrow} H_j(X^{(n)}, X^{(q)}) \)

Set \( C_n(X) = H_n(X^{(n)}, X^{(n-1)}) \) and define \( \partial_n : C_n(X) \rightarrow C_{n-1}(X) \) to be the boundary map of the triple \( (X^{(n)}, X^{(n-1)}, X^{(n-2)}) \) defined by:

\[
\begin{array}{ccc}
H_n(X^{(n)}, X^{(n-1)}) & \xrightarrow{\partial} & H_{n-1}(X^{(n-1)}) \\
\downarrow & & \downarrow \\
C_n(X) & \xrightarrow{\partial_n} & C_{n-1}(X).
\end{array}
\]

Then \( \partial_{n-1}\partial_n = 0 \) because

\[ H_{n-1}(X^{(n-1)}) \rightarrow H_{n-1}(X^{(n-1)}, X^{(n-2)}) \rightarrow H_{n-2}(X^{(n-2)}) \]

is exact.

\{C_n(X), \partial_n\} is called the cellular chain complex of X. A map \( f : X \rightarrow Y \) is cellular if \( f(X^{(n)}) \subset Y^{(n)} \). Such a map induces a chain map \( C_n(X) \rightarrow C_n(Y) \).

**Theorem.** \( H_n(C(X)) = H_n(X) \).

**Proof.** (1) \( H_n(X^{(p)}, X^{(q)}) = 0 \) for \( p \geq q \geq n \) or \( n > p \geq q \) by induction on \( p - q \).

This is trivial for \( p - q = 0 \). In the exact sequence of the triple

\[ H_n(X^{(q+1)}, X^{(q)}) \rightarrow H_n(X^{(p)}, X^{(q)}) \rightarrow H_n(X^{(p)}, X^{(q+1)}) \]

the first term is 0 by (1) and the last is 0 by induction.

(2) \( H_n(X, X^{(q)}) = 0 \) for \( q \geq n \) by (1) and (ii).

(3) \( H_n(X^{(q)}, X^{(r)}) = H_n(X, X^{(r)}) \) for \( q > n \) and \( q \geq r \) by (2) using the sequence of the triple \( X, X^{(q)}, X^{(r)} \).
(4) Chasing the diagram:
\[ H_{n+1}(X^{(n+1)}, X^{(n)}) \]
\[ \downarrow \partial \quad \nabla \phi_{n+1} \]
\[ 0 \rightarrow H_n(X^{(n)}) \rightarrow H_n(X^{(n)}, X^{(n-1)}) \rightarrow H_{n-1}(X^{(n-1)}) \]
\[ \downarrow \quad \nabla \phi_n \quad \downarrow \]
\[ H_n(X) = H_n(X^{(n+1)}) \]
\[ \downarrow \]
\[ 0 \]
yields \( H_n(X) = \ker \partial_n / \text{im} \partial_{n+1} \).

The \( n \)th group in the cellular chain complex,
\[ C_n(X) = \Sigma H_n(X^{(n)}, X^{(n-1)}) \]
\[ \overset{\Sigma F_\alpha}{\longrightarrow} \bigoplus_{\alpha} H_n(D^n, S^{n-1}) \]
can be regarded as the free abelian group generated by the \( n \)-dimensional cells of \( X \). We fix a consistent choice of generators using the sequence of isomorphisms from page 4:
\[ \tilde{H}_0(S^0) \leftarrow H_1(D^1, S^0) \rightarrow H_1(S^0) \leftarrow H_2(D^2, S^1) \rightarrow H_2(S^2) \leftarrow \cdots \]
taking generators corresponding to \((i_{1*} - i_{-1*})1 \in \tilde{H}_0(S^0)\).

The space \( X^{(n)}/X^{(n-1)} \cong \bigsqcup_{\alpha} S^n \) is homeomorphic to a bouquet of \( n \)-spheres, one for each \( n \)-cell in \( X \) and the composition
\[ (D^n, S^{n-1}) \overset{F_\beta}{\longrightarrow} (X^{(n)}, X^{(n-1)}) \rightarrow \bigsqcup_{\alpha} S^n \overset{r_\beta}{\longrightarrow} S^n \]
is the purse string map.

The boundary map \( \partial_n \) can be described in terms of the attaching maps \( f_\alpha \). Index the \( n \)-cells of \( X \) by \( \alpha \in A \) and the \( (n-1) \)-cells by \( \beta \in B \). Let \( c : X^{(n-1)} \rightarrow X^{(n-1)}/X^{(n-2)} \) be the collapsing map and \( \tau_\beta : X^{(n-1)}/X^{(n-2)} \rightarrow S^{n-1} \) retract to the \( \beta \)th cell modulo its boundary. Then the \( \alpha, \beta \) entry in the matrix representing \( \partial_n \) is \( \deg(\tau_\beta \circ c \circ f_\alpha) \).

\[ C_n(X) = H_n(X^{(n)}, X^{(n-1)}) \overset{\cong}{\leftarrow} \bigoplus_{\alpha} H_n(D^n, S^{n-1}) \]
\[ \downarrow \partial_n \quad \tilde{H}_{n-1}(X^{(n-1)}) \overset{\Sigma F_\alpha}{\longrightarrow} \bigoplus_{\alpha} \tilde{H}_0(S^{n-1}) \]
\[ \downarrow \quad \nabla \tilde{H}_{n-1}(X^{(n-2)}) \]
\[ C_{n-1}(X) = H_{n-1}(X^{(n-1)}, X^{(n-2)}) \overset{\cong}{\leftarrow} \bigoplus_{\beta} H_{n-1}(D^{n-1}, S^{n-2}) \]
\[ \downarrow \tilde{H}_{n-1}(X^{(n-2)}/X^{(n-2)}) \overset{\cong}{\leftarrow} \bigoplus_{\beta} \tilde{H}_{n-1}(S^{n-1}) \]
The real projective space $RP^n$ is obtained from $RP^{n-1}$ by attaching one cell of dimension $n$ by the double cover map $f_n : S^{n+1} \to RP^{n-1}$. Thus, by induction, $RP^n$ is a CW-space with one cell in each dimension up to $n$. The $n$-skeleton of $RP^m$ is $RP^n$. Simplifying the diagram above we have

\[
\begin{align*}
C_n(RP^m) & = H_n(RP^n, RP^{n-1}) \cong H_n(D^n, S^{n-1}) \\
& \downarrow a_n \quad \quad \downarrow \rho \\
H_{n-1}(RP^{n-1}) & \cong H_{n-1}(S^{n-1}) \quad = Z \\
& \downarrow f_n \quad \quad \quad \downarrow = \\
C_{n-1}(RP^m) & = H_{n-1}(RP^{n-1}, RP^{n-2}) \cong H_{n-1}(D^{n-1}, S^{n-2}) \quad = Z
\end{align*}
\]

The composition of the maps

\[
S^{n-1} \to RP^{n-1} \to (RP^{n-1}, RP^{n-2}) \cong (D^{n-1}, S^{n-2}) \\
RP^{n-1}/RP^{n-2} \quad = \quad D^{n-1}/S^{n-2} \quad = \quad S^{n-1}
\]

is

\[(1 \lor a)c : S^{n-1} \to S^{n-1}\]

which has degree $1 + (-1)^n$.

Thus the groups $C_n = Z$ and the chain complex is

\[
\cdots Z \xrightarrow{2} Z \xrightarrow{0} Z \xrightarrow{2} Z \xrightarrow{0} Z \to 0
\]

We find

\[
H_j(RP^n) = \begin{cases} 
Z & \text{for } j = 0, \\
Z/2Z & \text{for } 0 < j < n \text{ and } j \text{ odd}, \\
Z & \text{for } j = n \text{ if } n \text{ is odd}, \\
0 & \text{otherwise}
\end{cases}
\]

Complex projective space $CP^n = CP^{n-1} \cup_h D^{2n}$ attached by the Hopf map. Since $C_q(CP^n) = 0$ for $q$ odd, all $\partial_q$ are zero and

\[
H_q(CP^n) = \begin{cases} 
Z & \text{for } q \text{ even and } 0 \leq q \leq 2n, \\
0 & \text{otherwise}
\end{cases}
\]

Projective spaces and Hopf maps.

Let $F = R, C$, or the quaternions $H$. Let $d = \dim_R F$ which is 1, 2, or 4 respectively. $F^{n+1}$ is the vector space of $n + 1$-tuples, written $z = (z_0, \ldots, z_n)$, with inner product $z \cdot w = \sum z_j \overline{w}_j$. The projective space $FP^n$ is the space of lines in $F^{n+1}$; two nonzero
vectors $z$ and $w$ lie on the same line if and only if there is a nonzero scaler $\lambda \in F$ such that $w = \lambda z$.

The set of vectors in $F^{n+1}$ of length 1 is homeomorphic to $S^{dn+d-1}$. The Hopf map $h : S^{dn+d-1} \to F^p_n$ sends a nonzero vector $z$ to the line $[z]$ containing $z$. $h$ is onto and $f^{-1}([z]) \approx S^{d-1}$. Let

$$e_q = \{ [z] \in F^p_n : z_q \neq 0, z_j = 0 \text{ for } j > q \}$$

Then $F^p_n = \bigcup_{q=0}^n e_q$.

The unit disk in $F^q$ is homeomorphic to $D^d$. Define $G : D^d \to F^p_n$ by

$$G(z_0, \ldots, z_{q-1}) = [z_0, \ldots, z_{q-1}, r, 0, \ldots, 0] \text{ where } r = \sqrt{1 - z \cdot z}.$$  

The restriction of $G$ to the interior of the disk is a homeomorphism to the cell $e_q$ and the attaching map $G|_{\partial D^d} : S^{dq-1} \to F^p_{q-1}$ is the Hopf map, $h$. The projective line, $F^p_1 \approx S^d$. To see this define coordinate charts $f_1 : F \to F^p_1$ and $f_2 : F \to F^p_1$ by $f_1(z) = [z, 1]$ and $f_2(z) = [1, \bar{z}]$. Then $f_2^{-1}f_1(z) = \bar{z}^{-1}$ for $z \neq 0$. Regarding the nonzero elements of $F$ as $R^d - \{0\}$ the same coordinate change defines the manifold structure on $S^d$. Thus there are Hopf maps $S^3 \to S^2$ and $S^7 \to S^4$.

![Diagram](image-url)