

Homework #4

Due on 09/23/2026, Wednesday, before class.

1. Exercise 6.2.1 on page 373

6.2.1. Prove that $\bar{X}$, the mean of a random sample of size n from a distribution that is $N(\theta, \sigma^2)$, $-\infty < \theta < \infty$, is, for every known $\sigma^2 > 0$, an efficient estimator of θ .

2. Exercise 6.2.3 on page 374

6.2.3. Given the pdf

$$f(x; \theta) = \frac{1}{\pi[1 + (x - \theta)^2]}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty,$$

show that the Rao–Cramér lower bound is $2/n$, where n is the size of a random sample from this Cauchy distribution. What is the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta)$ if $\hat{\theta}$ is the mle of θ ?

3. Exercise 6.2.8 on page 375

6.2.8. Let X be $N(0, \theta)$, $0 < \theta < \infty$.

- (a) Find the Fisher information $I(\theta)$.
- (b) If $X_1, X_2, \dots, X_n$ is a random sample from this distribution, show that the mle of θ is an efficient estimator of θ .
- (c) What is the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta)$?

4. Exercise 6.2.12 on page 375 (for graduates only)

6.2.12. Recall that $\hat{\theta} = -n / \sum_{i=1}^n \log X_i$ is the mle of θ for a beta($\theta, 1$) distribution. Also, $W = -\sum_{i=1}^n \log X_i$ has the gamma distribution $\Gamma(n, 1/\theta)$.

- (a) Show that $2\theta W$ has a $\chi^2(2n)$ distribution.
- (b) Using part (a), find c_1 and c_2 so that

$$P\left(c_1 < \frac{2\theta n}{\hat{\theta}} < c_2\right) = 1 - \alpha, \quad (6.2.35)$$

for $0 < \alpha < 1$. Next, obtain a $(1 - \alpha)100\%$ confidence interval for θ .

- (c) For $\alpha = 0.05$ and $n = 10$, compare the length of this interval with the length of the interval found in Example 6.2.6.