

Contents

Preface	xi
1 Sets and Events	1
1.1 Introduction	1
1.2 Basic Set Theory	2
1.2.1 Indicator functions	5
1.3 Limits of Sets	6
1.4 Monotone Sequences	8
1.5 Set Operations and Closure	11
1.5.1 Examples	13
1.6 The σ -field Generated by a Given Class \mathcal{C}	15
1.7 Borel Sets on the Real Line	16
1.8 Comparing Borel Sets	18
1.9 Exercises	20
2 Probability Spaces	29
2.1 Basic Definitions and Properties	29
2.2 More on Closure	35
2.2.1 Dynkin's theorem	36
2.2.2 Proof of Dynkin's theorem	38
2.3 Two Constructions	40
2.4 Constructions of Probability Spaces	42
2.4.1 General Construction of a Probability Model	43
2.4.2 Proof of the Second Extension Theorem	49

2.5	Measure Constructions	57
2.5.1	Lebesgue Measure on $(0, 1]$	57
2.5.2	Construction of a Probability Measure on \mathbb{R} with Given Distribution Function $F(x)$	61
2.6	Exercises	63
3	Random Variables, Elements, and Measurable Maps	71
3.1	Inverse Maps	71
3.2	Measurable Maps, Random Elements, Induced Probability Measures	74
3.2.1	Composition	77
3.2.2	Random Elements of Metric Spaces	78
3.2.3	Measurability and Continuity	80
3.2.4	Measurability and Limits	81
3.3	σ -Fields Generated by Maps	83
3.4	Exercises	85
4	Independence	91
4.1	Basic Definitions	91
4.2	Independent Random Variables	93
4.3	Two Examples of Independence	95
4.3.1	Records, Ranks, Renyi Theorem	95
4.3.2	Dyadic Expansions of Uniform Random Numbers	98
4.4	More on Independence: Groupings	100
4.5	Independence, Zero-One Laws, Borel-Cantelli Lemma	102
4.5.1	Borel-Cantelli Lemma	102
4.5.2	Borel Zero-One Law	103
4.5.3	Kolmogorov Zero-One Law	107
4.6	Exercises	110
5	Integration and Expectation	117
5.1	Preparation for Integration	117
5.1.1	Simple Functions	117
5.1.2	Measurability and Simple Functions	118
5.2	Expectation and Integration	119
5.2.1	Expectation of Simple Functions	119
5.2.2	Extension of the Definition	122
5.2.3	Basic Properties of Expectation	123
5.3	Limits and Integrals	131
5.4	Indefinite Integrals	134
5.5	The Transformation Theorem and Densities	135
5.5.1	Expectation is Always an Integral on \mathbb{R}	137
5.5.2	Densities	139
5.6	The Riemann vs Lebesgue Integral	139
5.7	Product Spaces	143
5.8	Probability Measures on Product Spaces	147
5.9	Fubini's theorem	149
5.10	Exercises	155
6	Convergence Concepts	167
6.1	Almost Sure Convergence	167
6.2	Convergence in Probability	169
6.2.1	Statistical Terminology	170
6.3	Connections Between a.s. and i.p. Convergence	171
6.4	Quantile Estimation	178
6.5	L_p Convergence	180
6.5.1	Uniform Integrability	182
6.5.2	Interlude: A Review of Inequalities	186
6.6	More on L_p Convergence	189
6.7	Exercises	195
7	Laws of Large Numbers and Sums of Independent Random Variables	203
7.1	Truncation and Equivalence	203
7.2	A General Weak Law of Large Numbers	204
7.3	Almost Sure Convergence of Sums of Independent Random Variables	209
7.4	Strong Laws of Large Numbers	213
7.4.1	Two Examples	215
7.5	The Strong Law of Large Numbers for IID Sequences	219
7.5.1	Two Applications of the SLLN	222
7.6	The Kolmogorov Three Series Theorem	226
7.6.1	Necessity of the Kolmogorov Three Series Theorem	230
7.7	Exercises	234
8	Convergence in Distribution	247
8.1	Basic Definitions	247
8.2	Scheffé's lemma	252
8.2.1	Scheffé's lemma and Order Statistics	255
8.3	The Baby Skorohod Theorem	258
8.3.1	The Delta Method	261
8.4	Weak Convergence Equivalences; Portmanteau Theorem	263
8.5	More Relations Among Modes of Convergence	267
8.6	New Convergences from Old	268
8.6.1	Example: The Central Limit Theorem for m-Dependent Random Variables	270
8.7	The Convergence to Types Theorem	274
8.7.1	Application of Convergence to Types: Limit Distributions for Extremes	278
8.8	Exercises	282

9	Characteristic Functions and the Central Limit Theorem	293	
9.1	Review of Moment Generating Functions and the Central Limit Theorem	294	
9.2	Characteristic Functions: Definition and First Properties	295	
9.3	Expansions	297	
9.3.1	Expansion of e^{ix}	297	
9.4	Moments and Derivatives	301	
9.5	Two Big Theorems: Uniqueness and Continuity	302	
9.6	The Selection Theorem, Tightness, and Prohorov's theorem	307	
9.6.1	The Selection Theorem	307	
9.6.2	Tightness, Relative Compactness, and Prohorov's theorem	309	
9.6.3	Proof of the Continuity Theorem	311	
9.7	The Classical CLT for iid Random Variables	312	
9.8	The Lindeberg–Feller CLT	314	
9.9	Exercises	321	
10	Martingales	333	
10.1	Prelude to Conditional Expectation: The Radon–Nikodym Theorem	333	
10.2	Definition of Conditional Expectation	339	
10.3	Properties of Conditional Expectation	344	
10.4	Martingales	353	
10.5	Examples of Martingales	356	
10.6	Connections between Martingales and Submartingales	360	
10.6.1	Doob's Decomposition	360	
10.7	Stopping Times	363	
10.8	Positive Super Martingales	366	
10.8.1	Operations on Supermartingales	367	
10.8.2	Upcrossings	369	
10.8.3	Boundedness Properties	369	
10.8.4	Convergence of Positive Super Martingales	371	
10.8.5	Closure	374	
10.8.6	Stopping Supermartingales	377	
10.9	Examples	379	
10.9.1	Gambler's Ruin	379	
10.9.2	Branching Processes	380	
10.9.3	Some Differentiation Theory	382	
10.10	Martingale and Submartingale Convergence	386	
10.10.1	Krickeberg Decomposition	386	
10.10.2	Doob's (Sub)martingale Convergence Theorem	387	
10.11	Regularity and Closure	388	
10.12	Regularity and Stopping	390	
10.13	Stopping Theorems	392	
10.14	Wald's Identity and Random Walks	398	
10.14.1	The Basic Martingales	400	
10.14.2	Regular Stopping Times	402	
10.14.3	Examples of Integrable Stopping Times	407	
10.14.4	The Simple Random Walk	409	
10.15	Reversed Martingales	412	
10.16	Fundamental Theorems of Mathematical Finance	416	
10.16.1	A Simple Market Model	416	
10.16.2	Admissible Strategies and Arbitrage	419	
10.16.3	Arbitrage and Martingales	420	
10.16.4	Complete Markets	425	
10.16.5	Option Pricing	428	
10.17	Exercises	429	
	References	443	
	Index	445	