Stat 521 HW5 Fall 2012 Due October 12, 2012

Required Part:

0. Read §1f Extrema of Quadratic Forms, §3a Univariate Models, and §3b Sampling Distributions.

1. Let Σ be an $n \times n$ p.d. matrix. The inner product of $\mathbb{R}^n$ is defined by $(x, y) = x'\Sigma y$. Show that given an arbitrary $n \times n$ matrix A, an orthogonal projector onto $\mathcal{M}(A)$ is

$$P = A(A'\Sigma A)^{-}A'\Sigma.$$

Hint: See (vi) on page 47.

2. Let A be an $m \times m$ symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ and corresponding orthonormal eigenvectors $P_1, \ldots, P_m$. Consider k ($k \leq m$) mutually orthogonal vectors $\mathrm{x}_1, \ldots, \mathrm{x}_k$ in $\mathbb{R}^m$. Show that

$$\sup_{\mathrm{x}_1,\ldots,\mathrm{x}_k} \sum_{i=1}^{k} \frac{\mathrm{x}_i' A \mathrm{x}_i}{\mathrm{x_i}'\mathrm{x_i}} = \sum_{i=1}^{k} \lambda_i,$$

where the supremum is attained at $\mathrm{x}_i = cP_i$, $i = 1, \ldots, k$ for some scalar $c \neq 0$.

Hint: See (iv) on page 63.

3. Let A be an $n \times n$ symmetric matrix and B be an $m \times n$ matrix. Suppose a random vector $Y \sim N_n(\boldsymbol{\mu}, \sigma^2 I_n)$, where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)'$. Show that $Y'AY$ and BY are independent if $BA = 0$.

Hint: See Problem 1.2 on page 209.

4. Suppose $Y \sim N_n(X\beta, \sigma^2 I_n)$, where $Y = (y_1, \ldots, y_n)'$ is a random vector, X is an $n \times m$ matrix with rank r, and $\beta = (\beta_1, \ldots, \beta_m)'$ is a vector of parameters. Let $\hat{\beta} = (X'X)^{-}X'Y$ which is a solution to $X'X\beta = X'Y$. Let H be an $m \times k$ matrix, such that, $\mathcal{M}(H) = \mathcal{M}(X')$. Denote $Z = H'\hat{\beta}$ and $R_0^2 = (Y - X\hat{\beta})'(Y - X\hat{\beta})$.

(a) Show that $X = X(X'X)^{-}X'X$.

(b) Show that $I - X(X'X)^{-}X'$ is idempotent.

(c) Show that Z and R_0^2 are independent.

(d) Show that $Z \sim N_k(H'\beta, \sigma^2 H'(X'X)^{-}H)$, and $R_0^2 \sim \sigma^2\chi^2(n - r)$.

Hint: See Problems 2.1, 2.2, and 2.3 on page 210.

Optional Part:

5. Let A and B be $n \times n$ symmetric matrices. Suppose a random vector $Y \sim N_n(\boldsymbol{\mu}, \sigma^2 I_n)$, where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)'$. Show that $Y'AY$ and $Y'BY$ are independent if $AB = 0$ or $BA = 0$.

Hint: See Problem 1.1 on page 209.