Required Part:

0. Read §4h The General Problem of Least Squares with Two Sets of Parameters.

1. Consider a linear model

$$Y = X\beta + \epsilon \sim N_n(X\beta,\ \sigma^2 I_n),$$

where X is $n \times m$ with $R(X) = r < m \leq n$.

(a) Let $W = \{z \in \mathbb{R}^n \mid E(z'Y) = 0 \text{ for all } \beta\}$. Show that W is a linear subspace with dimension $n - r$.

(b) Let $\{z_1, \ldots, z_{n-r}\}$ be an orthonormal basis for W. Show that

$$R_0^2 = \min_{\beta}(Y - X\beta)'(Y - X\beta) = \sum_{i=1}^{n-r}(z_i'Y)^2.$$

(c) Use the result in (b) to construct an unbiased estimator for σ^2 as a function of $z_1'Y, \ldots, z_{n-r}'Y$.

(d) Suppose $Y = (Y_1, \ldots, Y_n)'$ with $E(Y_1) = E(Y_2)$. Let

$$p'\hat{\beta} = b_1 Y_1 + b_2 Y_2 + \cdots + b_n Y_n$$

is the BLUE of an estimable function $p'\beta$, show that $b_1 = b_2$.

2. Consider a linear model

$$Y = X\beta + \epsilon \sim N_n(X\beta,\ \sigma^2 \Sigma),$$

where X is $n \times m$ and Σ is $n \times n$. Suppose $R(\Sigma) = k < n$.

(a) Show that there are $n - k$ linearly independent vectors $w_1, \ldots, w_{n-k}$ in $\mathbb{R}^n$ such that

$$P(w_i'Y = w_i'X\beta) = 1 \text{ for each } i = 1, \ldots, n - k.$$

(b) Show that there exists an $n \times k$ matrix H of rank k such that

$$Z = H'Y \sim N_k(H'X\beta,\ \sigma^2 I_k).$$

3. Let $\{Y_{ij},\ i = 1, \ldots, p;\ j = 1, \ldots, q\}$ be independent random variables such that $E(Y_{ij}) = 0$ and $V(Y_{ij}) = \sigma^2$. Show that

$$E\left[\sum_{i=1}^{p}\sum_{j=1}^{q}\left(Y_{ij} - \bar{Y}_{i\bullet} - \bar{Y}_{\bullet j} + \bar{Y}_{\bullet\bullet}\right)^2\right] = (p-1)(q-1)\sigma^2,$$

where $\bar{Y}_{i\bullet} = \sum_{j=1}^{q} Y_{ij}/q$, $\bar{Y}_{\bullet j} = \sum_{i=1}^{p} Y_{ij}/p$, $\bar{Y}_{\bullet\bullet} = \sum_{i=1}^{p}\sum_{j=1}^{q} Y_{ij}/(pq)$.