1. We consider the null hypothesis $H_0 : \mathbf{C}\boldsymbol{\beta} = \boldsymbol{\Gamma}_0$, where $\mathbf{C}$ is an $(r-q)\times(r+1)$ known matrix with rank $r-q$, $\boldsymbol{\beta}$ is an $(r+1)\times m$ matrix of unknown parameters, and $\boldsymbol{\Gamma}_0$ is an $(r-q)\times m$ known matrix. Show that there always exists an $(r+1)\times m$ matrix $\boldsymbol{\beta}_0$ satisfying $\mathbf{C}\boldsymbol{\beta}_0 = \boldsymbol{\Gamma}_0$ and an $(r+1)\times(q+1)$ full-rank matrix $\mathbf{B}$ satisfying $\mathbf{CB} = 0$, such that, $\mathbf{C}\boldsymbol{\beta} = \boldsymbol{\Gamma}_0$ if and only if $\boldsymbol{\beta} = \boldsymbol{\beta}_0 + \mathbf{B}\boldsymbol{\theta}$ for some $(q+1)\times m$ matrix $\boldsymbol{\theta}$. In other words,

$$\{\boldsymbol{\beta} \in \mathbb{R}^{(r+1)\times m} \mid \mathbf{C}\boldsymbol{\beta} = \boldsymbol{\Gamma}_0\} = \{\boldsymbol{\beta}_0 + \mathbf{B}\boldsymbol{\theta} \mid \boldsymbol{\theta} \in \mathbb{R}^{(q+1)\times m}\}$$

2. Predictions from multivariate multiple regressions Consider the multivariate linear regression model $\mathbf{Y} = \mathbf{Z}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where $\mathbf{Y}$ is an $n\times m$ matrix of responses, $\mathbf{Z}$ is an $n\times(r+1)$ known matrix with rank $(r+1)$, $\boldsymbol{\beta}$ is an $(r+1)\times m$ matrix of parameters, and $\boldsymbol{\varepsilon} = (\boldsymbol{\varepsilon}_1, \dots, \boldsymbol{\varepsilon}_n)'$ is an $n\times m$ matrix of random errors where $\boldsymbol{\varepsilon}_1, \dots, \boldsymbol{\varepsilon}_n$ are i.i.d.$\sim N_m(0, \boldsymbol{\Sigma})$. Let $\hat{\boldsymbol{\beta}} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Y}$, $\hat{\boldsymbol{\Sigma}} = \frac{1}{n}(\mathbf{Y}-\mathbf{Z}\hat{\boldsymbol{\beta}})'(\mathbf{Y}-\mathbf{Z}\hat{\boldsymbol{\beta}})$ be the MLE estimator of $\boldsymbol{\beta}$ and $\boldsymbol{\Sigma}$, respectively. We use $\hat{\boldsymbol{\beta}}'\mathbf{z}_0$ to estimate the response vector $\mathbf{Y}_0$ at a fixed predictor vector $\mathbf{z}_0 \in \mathbb{R}^{r+1}$.

(1) Show that $\hat{\boldsymbol{\beta}}'\mathbf{z}_0 \sim N_m(\boldsymbol{\beta}'\mathbf{z}_0,\ \mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0\boldsymbol{\Sigma})$.

(2) Show that $\hat{\boldsymbol{\beta}}'\mathbf{z}_0$ is independent of $n\hat{\boldsymbol{\Sigma}}$.

(3) Let

$$T^2 = \left(\frac{\hat{\boldsymbol{\beta}}'\mathbf{z}_0 - \boldsymbol{\beta}'\mathbf{z}_0}{\sqrt{\mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0}}\right)' \left(\frac{n\hat{\boldsymbol{\Sigma}}}{n-r-1}\right)^{-1} \left(\frac{\hat{\boldsymbol{\beta}}'\mathbf{z}_0 - \boldsymbol{\beta}'\mathbf{z}_0}{\sqrt{\mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0}}\right)$$

Show that $T^2 \sim \frac{m(n-r-1)}{n-r-m} F_{m,n-r-m}$.

(4) Show that the prediction error $\mathbf{Y}_0 - \hat{\boldsymbol{\beta}}'\mathbf{z}_0 \sim N_m(0,\ (1+\mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0)\boldsymbol{\Sigma})$ and

$$\left(\frac{\mathbf{Y}_0 - \hat{\boldsymbol{\beta}}'\mathbf{z}_0}{\sqrt{1+\mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0}}\right)' \left(\frac{n\hat{\boldsymbol{\Sigma}}}{n-r-1}\right)^{-1} \left(\frac{\mathbf{Y}_0 - \hat{\boldsymbol{\beta}}'\mathbf{z}_0}{\sqrt{1+\mathbf{z}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{z}_0}}\right) \sim \frac{m(n-r-1)}{n-r-m} F_{m,n-r-m}$$