I. **Review quadratic +exp**  

\[ ax^2 + bx + c = 0, \quad a \neq 0 \text{ and } a > 0. \]

\[ x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \]

\[ (x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2) - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0 \]

\[ (x + \frac{b}{2a})^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0 \]

\[ (x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2} \]

\[ x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a} \]

\[ x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \]

**Real solutions** when \( b^2 - 4ac \geq 0 \).

**Complex number (imaginary) solutions** when \( b^2 - 4ac < 0 \).

**Simplest complex solve**:

\[ x^2 + 1 = 0 \]

\[ x = \left(-1 \pm \sqrt{0^2 - 4(1)(1)}\right)/2 \]

\[ x = \pm \sqrt{-1} \]

There is no real number \( \sqrt{-1} \).

Initially people dealt with \( \sqrt{-1} \) by writing \( i = \sqrt{-1} \), \( -i = -\sqrt{-1} \) and using the rule \( i^2 = -1 \).
Then, assuming $i$ behaves otherwise as an ordinary number, we have
\[(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i\]

This actually gives us a rigorous definition of complex numbers. (Due to W.R. Hamilton in 1800's). We identify $a + bi = (a, b)$ as an ordered pair of real numbers. Thus $1 = (1, 0)$

and define the product of these pairs by: $(a, b)(c, d) = (ac - bd, ad + bc)$

and the sum by: $(a, b) + (c, d) = (a + c, b + d)$.

In this way we have $(a, b) = (a', b')$ if and only if $a = a'$ and $b = b'$.

Note that $(a, b)i = (-b, a)$.

If you think of $(a, b)$ as a point in the plane, then we have a geometric interpretation of complex numbers.

![Diagram](image-url)
Multiplication by $i$ rotates a point in the plane by $\frac{\pi}{2} = 90^\circ$.

\[
\begin{align*}
\hat{z}^2 &= -1 \\
\text{means that} & \\
\text{two } 90^\circ \text{ rotations} & \\
\text{make a } 180^\circ \text{ rotation!}
\end{align*}
\]

**Theorem.** Let $Z_\theta = \cos(\theta) + i \sin(\theta)$.

Then $Z_\theta Z_\phi = Z_{(\theta + \phi)}$.

**Proof.**

\[
Z_\theta Z_\phi = [\cos(\theta) + i \sin(\theta)] [\cos(\phi) + i \sin(\phi)]
\]

\[
= [\cos(\theta) \cos(\phi) - \sin(\theta) \sin(\phi)] + i [\sin(\theta) \cos(\phi) + \cos(\theta) \sin(\phi)]
\]

\[
= \cos(\theta + \phi) + i \sin(\theta + \phi)
\]

\[
= Z_{(\theta + \phi)} \quad //
\]

\begin{align*}
\sin(\theta) &= \sin(\phi) \\
\cos(\phi) &= \cos(\phi)
\end{align*}

We use basic trig formulas for sine and cosine of sums of angles.
Any complex number can be written in the form \( z = \rho z_\theta \) for some \( \rho \geq 0 \) and \( \theta \) an angle between \( 0 \neq 2\pi \).

Thus, when you multiply complex nos., you add their angles and multiply their lengths.

\[(\rho z_\theta)(\rho' z_{\theta'})) = (\rho \rho') z_{(\theta + \theta')}\]

\[\mathbb{C} = \{a+bi \mid a, b \in \mathbb{R}, \mathbb{R} = \text{the real numbers}\}\]

Example. Find all \( z \in \mathbb{C} \) s.t. \( z^3 = 1 \).

Answer. Let \( z = z_\theta \) then \( 3\theta = 2\pi N \)

So \( \theta = \frac{2\pi}{3} \). Thus we can have

\[\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \phi:\]

\[
\begin{align*}
Z_{\frac{2\pi}{3}} &= -\frac{1}{2} + \frac{\sqrt{3}}{2} i \\
Z_{\frac{4\pi}{3}} &= -\frac{1}{2} - \frac{\sqrt{3}}{2} i \\
Z_{\phi} &= 1 = 1
\end{align*}
\]

\(1, w, w^2\) are the three cube roots of unity.
Some facts:

1. \[ z = a + bi \]
   \[ \bar{z} = a - bi \]
   \[ \bar{z} \bar{z} = (a + bi)(a - bi) = a^2 + b^2 = ||z||^2 \]
   \[ ||z|| = \sqrt{a^2 + b^2} = \sqrt{z \bar{z}}. \]

   \[ ||z||^2 = a^2 + b^2. \]

2. \[ \overline{z \bar{w}} = \overline{z} \overline{\bar{w}}. \]
   Pf. \[ \overline{z \bar{w}} = \overline{(a + bi)(c + di)} \]
   \[ = (ac - bd) + (ad + bc)i \]
   \[ = (ac - bd) - (ad + bc)i \]
   \[ \overline{z \bar{w}} = (a - bi)(c - di) \]
   \[ = (ac - bd) - (bc - ad)i \]

3. Note:
   \[ (a^2 + b^2)(c^2 + d^2) = z\bar{z} \bar{w}\overline{\bar{w}} \]
   \[ = (\bar{z}w)\overline{\bar{w}} \]
   \[ = (\bar{z}w)(\overline{\bar{w}}) \]
   \[ (a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2 \]

   Thus the product of two sums of squares is a sum of squares. (Try \((1^2 + 2^2)(3^2 + 4^2)\).)
4.0 Solving cubic equations

Fact: 
\[(a+b)^3 = a^3 + b^3 + 3ab^2 + 3a^2b\]

So 
\[(a+b)^3 = (3ab)(a+b) + (a^3 + b^3)\].

If you were asked to solve 
\[X^3 = pX + q\]
you can set \[x = a+b\] (\(a, b\) unknown) and try to find \(a, b\) s.t.

\[\begin{align*}
p &= 3ab \\
q &= a^3 + b^3
\end{align*}\]

This leads to a quadratic equation

for \(a^3 = R\) and \(b^3 = S\).

\[(p/3)^3 = a^3b^3 = RS\]

\[q = a^3 + b^3 = R + S\].

Solve for \(R\) and \(S\).

Then take their cube roots and add them up.

Example: 
\[X^3 = 3X + 1\].

\[p = 3, \ q = 1\]

\[\begin{align*}
3 &= 3ab \\
1 &= a^3 + b^3
\end{align*}\]

So 
\[1 = ab\]
\[1 = a^3 + b^3\]

So 
\[1 = a^3b^3 = RS\]
\[1 = a^3 + b^3 = R + S\].

\[\text{next page}\]
\[
1 = RS \\
1 = R + S
\]

\[
S = 1 - R \\
1 = RS = R(1 - R)
\]

\[
1 = R - R^2 \\
R^2 - R + 1 = 0
\]

\[
R = \frac{1 \pm \sqrt{1 - 4}}{2} = \frac{1 \pm \sqrt{3}i}{2}.
\]

This means we can take

\[
a^3 = R = \frac{1 + \sqrt{3}i}{2} = \cos(60^\circ) + i\sin(60^\circ)
\]

\[
b^3 = S = 1 - R = \frac{1 - \sqrt{3}i}{2}.
\]

So we can take

\[
a = \sqrt{\frac{1 + \sqrt{3}i}{2}}, \quad b = \sqrt{\frac{1 - \sqrt{3}i}{2}}.
\]

These are the "principal cube roots of \(R\) and \(S\)."
Note that $x = a + b$ is real.

Thus we have shown that $x^3 = 3x + 1$ has a real root $x = 2 \cos(20^\circ) = \sqrt{\frac{1 + \sqrt{3}i}{2}} + \sqrt{\frac{1 - \sqrt{3}i}{2}}$.

What about the other roots of $x^3 = 3x + 1$?

Recall our 3 cube roots of unity $\xi, \omega, \omega^2$. We have

$$ab = \sqrt[3]{R} \sqrt[3]{S} = \sqrt[3]{\frac{(1+\sqrt{3}i)}{2}} \frac{(1-\sqrt{3}i)}{2} = \sqrt[3]{1} = 1.$$ 

If we replace $a$ by $wa = a'$

$$b$$ by $\omega^2 b = b'$,

then $a'b' = waw^2b = w^3ab = 1$.

So we still have

$$P = 3 a'b'$$

$$\Phi = a'^3 + b'^3.$$ 

Thus we have found the 3 roots to the cubic:

$$a + b,$$

$$wa + w^2b,$$

$$w^2a + wb.$$
Are the two new roots complex or real?

A complex number is real if and only if \( z = \overline{z} \).

\[
\begin{align*}
&z = a + bi; z = \overline{z} \iff a + bi = a - bi \\
&\iff a = a \text{ and } b = -b \iff b = 0 \end{align*}
\]

Now \( a + b = a + \overline{a} \) \( (b = a) \).

So \( \overline{a + b} = \overline{a + \overline{a}} = \overline{a + a} = \overline{a} + a = a + \overline{b} \).

Note:\( \overline{w} = \omega^2 \)

\( \frac{\omega}{\omega^2} = \omega \)

\( w \overline{a + \omega^2 b} = w \overline{a} + w^2 \overline{b} \)

\( = \omega^2 b + \omega a = \omega a + \omega^2 b. \)

Thus the other two roots of \( x^3 = 3x + 1 \) are also real!

and \( w \) \( x^3 = 3x + 1 \) has three real roots. If you plot \( y = x^3 - 3x - 1 \) it will look like:
Here is another example.

Consider the cubic
\[ x^3 = -3x + 1 \]

Then with \( x = a + b \), we have

\[ \begin{align*}
  +3ab &= -3 \\
  a^3 + b^3 &= 1
\end{align*} \]

\[ \begin{align*}
  ab &= -1 \\
  a^3 + b^3 &= 1
\end{align*} \]

\[ \Rightarrow \]

\[ \begin{align*}
  a^3b^3 &= -1 \\
  a^3 + b^3 &= 1
\end{align*} \]

So

\[ \begin{align*}
  RS &= -1 \\
  R + S &= 1
\end{align*} \]

\[ \begin{align*}
  R(1 - R) &= -1 \\
  R - R^2 &= -1 \\
  R^2 - R - 1 &= 0
\end{align*} \]

\[ R = \frac{1 \pm \sqrt{5}}{2} \quad \text{The quadratic has real roots.} \]

So

\[ a^3 = \frac{1 + \sqrt{5}}{2}, \quad b^3 = \frac{1 - \sqrt{5}}{2} \]

and the full set of roots to the cubic is : (next page)
Let \( a = \sqrt[3]{\frac{1 + \sqrt{5}}{2}} \), \( b = \sqrt[3]{\frac{1 - \sqrt{5}}{2}} \).

These are real numbers, \( a > 0 \), \( b < 0 \).

Then solve to \( x^3 = -3x + 1 \)

are: \( a + b \) (real) \[ \begin{cases} wa + w^2 b \\ w^2 a + wb \end{cases} \] both complex.

Note that \[ \frac{wa + w^2 b}{w a + w^2 b} = \frac{wa + w^2 b}{w^2 a + wb} = w^2 a + wb. \]

Thus the two complex roots are conjugates of each other.

\[ y = x^3 + 3x - 1 \]
II. *Induction*

**Principle of Mathematical Induction**

Suppose that $P(n)$ is a statement about a natural number $n$. Let $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$. Then $P(n)$ is true for all $n \in \mathbb{N}$ if you prove:

1. $P(1)$ is true.

and II. If $P(k)$ is true (for some $k$) then $P(k+1)$ is true.

That is, you must show that $P(k) \Rightarrow P(k+1)$.

**Example.** Show that $1 + 3 + 5 + \ldots + (2n-1) = n^2$ for all $n = 1, 2, 3, \ldots$.

**Solution.**

1. **$P(n)$: $1 + 3 + \ldots + (2n-1) = n^2$**

   $P(1): 1 = 1^2$ ($2\cdot 1 - 1 = 1$)

   $\therefore P(1)$ is true.

2. Suppose $1 + 3 + \ldots + (2k-1) = k^2$ (i.e. assume $P(k)$ is true).

   Then $1 + 3 + \ldots + (2k-1) + (2(k+1)-1)$

   $= k^2 + 2(k+1) - 1$

   $= k^2 + 2k + 1$

   $= (k+1)^2$. Thus we showed that $P(k) \Rightarrow P(k+1)$. \(\square\)
**Strong Induction.** For part II you can assume that $P(1), P(2), \ldots, P(k)$ are all true and prove from that that $P(k+1)$ is true.

It is not hard to show that Induction and Strong Induction are logically equivalent.

**Example.**

**Theorem.** Let $S \subseteq N = \{1, 2, 3, \ldots\}$ be a non-empty subset of the natural numbers $N$. Then $S$ has a least member.

**Proof.** Let $P(n) : n \notin S$.

Suppose that $S$ has no least member and that $S \subseteq N$. We will prove, by strong induction, that $S$ is empty.

I. $1 \notin S$ since $1 \in S \Rightarrow S$ has a least member.

II. Suppose that $1 \notin S, 2 \notin S, \ldots, k \notin S$. Then clearly if $(k+1) \in S$, $k+1$ would be the least member of $S$.

\[ \therefore \] $k+1 \notin S$.

We have proved, by induction, that $S$ is empty. \text{ i.e. } $S$ not empty implies $S$ has a least member. //
### III. Multiplying Permutations

The permutations of $1, 2, 3$ are:

$$
\begin{align*}
1 & 2 & 3 \\
1 & 3 & 2 \\
2 & 1 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2 \\
3 & 2 & 1 \\
\end{align*}
$$

These are all the ways to arrange three things $(1, 2, 3)$ in order.

I will write $(1\ 2\ 3)$ to denote a given permutation. Thus $(1\ 2\ 3)$ denotes "231".

I think of this as a mapping from the set $\{1, 2, 3\}$ to itself.

I have drawn it as:

$$
\begin{align*}
1 & \rightarrow 1 \\
2 & \rightarrow 2 \\
3 & \rightarrow 3 \\
\end{align*}
$$

or

$$
\begin{align*}
1 & \rightarrow 2 \\
2 & \rightarrow 3 \\
3 & \rightarrow 1 \\
\end{align*}
$$

\[ f: \{1, 2, 3\} \rightarrow \{1, 2, 3\} \]

\( (1)f = 2 \)
\( (2)f = 3 \)
\( (3)f = 1 \)

Notice I write \( f(x) \) instead of \( f(x) ! \)
Suppose

\[ f \quad \\begin{array}{ccc} \times & 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 \end{array} \quad \begin{array}{ccc} \times & 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 \end{array} \quad g \]

\[ (x) f g = \text{result of} \quad \begin{array}{ccc} \times & 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 \end{array} \quad \begin{array}{ccc} \times & 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 \end{array} \quad \text{doing } f \quad \text{then} \quad \text{doing } g. \]

e.g. \( (1) f g = (2) g = 1. \)

We can diagram it:

\[ f \quad \begin{array}{ccc} \times & 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 3 \end{array} \quad g \quad \begin{array}{ccc} \times & 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 \end{array} \]

I will eliminate the arrows and write

\[ \begin{array}{ccc} \times & 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 \end{array} \quad f \quad \begin{array}{ccc} \times & 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 \end{array} \quad h = fg. \]
So you see, you can multiply permutations.

e.g. \( R = \begin{array}{ccc}
\text{x} & \text{x} & \text{x} \\
\text{x} & \text{x} & \text{x} \\
\text{x} & \text{x} & \text{x} \\
\end{array} \)

\[ R^2 = \begin{array}{ccc}
\text{x} & \text{x} & \text{x} \\
\text{x} & \text{x} & \text{x} \\
\text{x} & \text{x} & \text{x} \\
\end{array} \quad = \quad \begin{array}{ccc}
\text{x} & \text{x} & \text{x} \\
\text{x} & \text{x} & \text{x} \\
\text{x} & \text{x} & \text{x} \\
\end{array} \]

Simplify and redraw the connections. Only endpoints matter.

\[ R^3 = \begin{array}{ccc}
\text{x} & \text{x} & \text{x} \\
\text{x} & \text{x} & \text{x} \\
\text{x} & \text{x} & \text{x} \\
\end{array} \quad = \quad \begin{array}{ccc}
\text{I} \\
\text{I} \\
\text{I} \\
\end{array} \quad = \quad \text{I} \]

We have 6 permutations.

\( \text{III} \times \text{X} \times \text{IX} \times \text{XX} \times \text{I} \)

\( \text{I} \times R \times R^2 \times F_1 \times F_2 \times F_3 \)

You can make a 6x6 multiplication table.

(Homework.)
Homework

1. Draw the full architecture for 
\[(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\]
Show how a cube of side \(a+b\) decomposes into 3 dim rectangular parts in correspondence with the above formula.

2. Prove by induction that
\[(1^3 + 2^3 + \ldots + n^3) = (1+2+\ldots+n)^2\]
for all \(n = 1, 2, 3, \ldots\)

3. Prove by induction (strong induction) that every natural number can be written as a sum of distinct powers of 2.
(e.g. 27 = 2^4 + 2^3 + 2^1 + 2^0)

4. Using numbers different from our examples, solve a cubic of the form \(x^3 = px + q\)
(P and q real numbers that you choose.)

5. Make a multiplication table for \(\mathbb{E}, \mathbb{R}, \mathbb{R^2}, \mathbb{F}_1, \mathbb{F}_2, \mathbb{F}_3\) as on page 16.