Assignment for April 29, 2014.

- Read these notes and compare with (a) the notes on finite subgroups of SO(3).
  (b) Goodman, Chapter 5.

- <u>Goodman, Chapter 5</u>

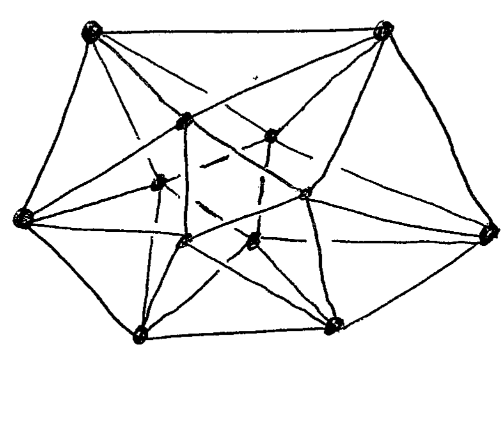
  <u>Exercises 5.1, P. 247 →</u>

  5.1.1, 5.1.3, 5.1.5, 5.1.20

  <u>Exercises 5.2, P. 252 →</u>

  5.2.3, 5.2.4.

In the notes on Classifying Finite Subgroups of SO(3), you will find that Theorem 7.3 is the Burnside Theorem of these notes. So you can go ahead and read section 8. Please do so.



Algebra Notes #10 — Math 435, Spring 2014

| See page ● for your assignment. |

1° Cosets

$H \subseteq G$ a subgroup of the group G.

$g \in G$, $gH \underset{def}{=} \{gh \mid h \in H\}$

gH is called a (left) coset of H.

(We can also form $Hg = \{hg \mid h \in H\}$, a right coset.)

Note: $\boxed{gH = g'H \iff g^{-1}g' \in H.}$

Pf. If $g^{-1}g' \in H$ then $g^{-1}g'H = H$.

(i.e. $h \in H \Rightarrow hH = H$ since $\{hh' \mid h' \in H\} = H$ (exercise!))

Thus $g^{-1}g'H = H \Rightarrow g(g^{-1}g')H = gH$

$\Longrightarrow g'H = gH$. ✓

If $gH = g'H$ then $g^{-1}(gH) = g^{-1}(g'H)$

$\Rightarrow (g^{-1}g)H = (g^{-1}g')H$ (Exercise: prove $(ab)H = a(bH)$ $\forall a, b \in G$.)

Thus $H = (g^{-1}g')H$ ✓

$\therefore g^{-1}g' \in H$. //

Corollary. $(gH) \cap (g'H) = gH$ or $\emptyset$.

That is, two cosets are either equal or disjoint.

Pf. Suppose $x \in gH \cap g'H$. Then $x = gh$, $h \in H$ & $x = g'h'$, $h' \in H$.

$\therefore gh = g'h' \Rightarrow g^{-1}g' = hh'^{-1} \in H$

$\Longrightarrow gH = g'H$. //

Let G be a finite group.
Then $\{gH \mid g \in G\}$ is finite & so
$\exists\ g_1, g_2, \ldots, g_K \in G$, distinct elements of G
such that $\{gH \mid g \in G\} = \{g_1H, g_2H, \ldots, g_KH\}$
and $g_iH \cap g_jH = \phi$ if $i \neq j$.
Thus $G = (g_1H) \cup (g_2H) \cup \cdots \cup (g_KH)$.
G is a disjoint union of K cosets of H. $\therefore$ $\boxed{\#G = K(\#H)}$ (since each coset satis $\#(gH) = \#(H)$.)

We have proved that

Proposition. If G is a finite group and H is a subgroup of G, then the order of H divides the order of G: $\#(H) \mid \#(G)$.

Corollary. Let $a \in G$, G a finite group. Then $\text{order}(a) \mid \#(G)$.

Pf. $\text{order}(a) =$ least $n > 0$ s.t. $a^n = e$.
Let $H = \{e, a, a^2, \ldots, a^{n-1}\}$. $\#H = n$
$H \subseteq G$ subgroup. $\therefore$ $\#H \mid \#G$.
$\therefore$ $n = \text{order}(a) \mid \#(G)$. //

Remark. We have seen many examples of this. e.g. $\text{order}(2) = 12$ in $\mathcal{U}_{35}$ & $\#\mathcal{U}_{35} = 24$.

More generally, we know that $\# \mathcal{U}_n = \phi(n)$ where $\phi(n)$ is the Euler ϕ-function. Thus we have that if $a \in \mathcal{U}_n$ then $a^{\phi(n)} \equiv 1$ (modulo n).

(Exercise: Show that for any $a \in \mathbb{Z}_n$, $a^{\phi(n)} \equiv 1$ (modulo n).)

Note: $g \in G$, $H \subseteq G$ subgroup, then $gHg^{-1} = \{ghg^{-1} \mid h \in H\}$ is a subgroup of G. gHg^{-1} is called a conjugate subgroup to H.

$H \subseteq G$ is called normal (we write $H \triangleleft G$) if $gHg^{-1} = H \quad \forall g \in G$.

Let $G/H = \{gH \mid g \in G\}$ = set of all left cosets of H.

Define a product for cosets:

$$(gH)(g'H) = (gg')H.$$

Claim. This definition is well-defined.

Pf: Must show $g_1H = g_2H \Rightarrow (g_1H)(g'H) = (g_2H)(g'H)$.

$(g_1H)(g'H) = g_1g'H$
$(g_2H)(g'H) = g_2g'H$

But $g'Hg'^{-1} = H$ ($H \triangleleft G$) $\therefore g'H = Hg'$

$\therefore g_1g'H = g_1(g'H) = g_1Hg' = (g_1H)g' = (g_2H)g' = g_2(Hg') = g_2(g'H) = (g_2g')H.$

Given $H \triangleleft G$ we have a well-defined product of cosets:

$$(gH)(g'H) = (gg')H.$$

Note $H(g'H) = g'H \quad \forall g' \in G$

So H acts as the identity.

$$(g^{-1}H)(gH) = (g^{-1}g)H = eH = H.$$

Thus we have inverses. Associativity is straightforward.

<u>Thus</u>: If $H \triangleleft G$, then G/H is a group.

The set of cosets of H forms a group when H is a normal subgroup of G.

2°. <u>Group Actions</u> [See Goodman, Chapter 5]

An <u>action</u> of G, a group, on a set X is a mapping $G \xrightarrow{F} Symm(X)$ that is a <u>homomorphism of groups</u>, where $Symm(X)$ denotes the set of 1-1, onto maps of X (which is a group under composition). Recall that $Symm\{1, 2, \ldots, n\} = S_n$, the symmetric group on n letters.

Thus $g \in G$, then

$F(g) : X \longrightarrow X$, a 1-1 onto mapping of X to itself. And if $g' \in G$, then $F(g) \circ F(g') = F(gg')$.

If $F : G \longrightarrow \mathrm{Symm}(X)$ is an action of G on X, then we can define $\hat{F} : G \times X \longrightarrow X$ by

$$\hat{F}(g, x) = F(g)(x).$$

Given $g \in G$ and $x \in X$ we write

$$gx \underset{def}{=} \hat{F}(g,x) = F(g)(x).$$

$$G \times X \longrightarrow X$$
$$g, x \longmapsto gx.$$

Note $\quad g_1(g_2 x) = (g_1 g_2)x$

& $\quad ex = x.$

<u>Orbits</u>: We say $x \sim y$ if $\exists g,\ gx = y$.

$$\overset{\circ}{x} \xrightarrow{\quad g \quad} \overset{\circ}{y} = gx$$

The <u>orbit of x</u> $= \{gx \mid g \in G\}$

$\qquad \| \qquad\qquad\qquad\qquad \|$

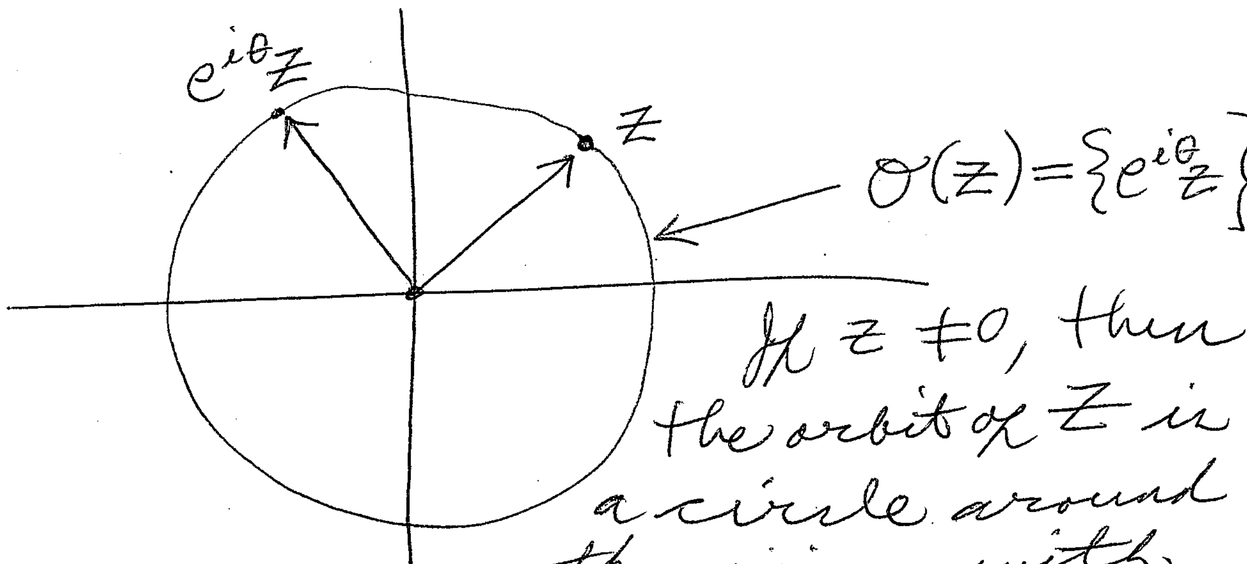
$\quad \mathcal{O}(x) \qquad\qquad\qquad\quad Gx$

$$\boxed{\mathcal{O}(x) = Gx}$$

Example. $G = S^1 = \{e^{i\theta}\}$ unit complex nos.

$X = \mathbb{C} = \{a+bi \mid a, b \text{ real}\}$ complex plane.

$$S^1 \times X \longrightarrow X$$
$$e^{i\theta}, z \longmapsto e^{i\theta} z \quad (\text{multiply})$$



If $z \neq 0$, then the orbit of z is a circle around the origin with radius $= |z|$.

If $z = 0$, then $\mathcal{O}(0) = \{0\}$.

Stabilizer Given $x \in X$, let

$\text{Stabilizer}(x) = \{g \in G \mid gx = x\}$.

||| notation subgroup of G!

$\text{Stab}(x)$.

Prop: $G/\text{Stab}(x) \longleftrightarrow \mathcal{O}(x)$.

Pf: $g\,\text{Stab}(x) \longleftrightarrow gx$.
check that this is a 1-1 corres.

<u>Corollary</u>. If G is finite, then

$$\#\mathcal{O}(x) = \#(G/\mathrm{Stab}(x))$$
$$= \#(G)/\#(\mathrm{Stab}(x))$$

(by our results on cosets)

$\therefore$ $\underline{\#\mathcal{O}(x) \times \#\mathrm{Stab}(x) = \#(G)}$.

and $\#\mathcal{O}(x) \,|\, \#(G)$.

<u>Theorem</u>. Let p be a prime number, G a finite group. Suppose $p \,|\, \#(G)$. Then $\exists g \in G$ with $\mathrm{order}(g) = p$.

<u>Proof</u>. Let $X = \left\{ (a_1, a_2, \ldots, a_p) \,\middle|\, \begin{array}{l} a_1 a_2 \cdots a_p = e, \\ a_i \in G, i = 1, \ldots, p \end{array} \right\}$

$\#X = (\#G)^{p-1}$ (since $a_p = a_{p-1}^{-1} a_{p-2}^{-1} \cdots a_2^{-1} a_1^{-1}$)

Since $ab = e \Rightarrow ba = e$ [$(ba)b = b(ab) = be = b \Rightarrow ba = e$]

we have $(a_1, \ldots, a_p) \in X \Rightarrow (a_p, a_1, \ldots, a_{p-1}) \in X$.

So cyclic group C_p acts on X.

Each $x \in X$ is either a fixed point or has an orbit of size p.

$\therefore \#X = \sum \#\mathrm{Orbits} = n + kp$ (n = # fixed pts; k = # orbits of size p.)

$\therefore p \,|\, \#X - (kp)$ (since $p \,|\, \#X$)

$\therefore p \,|\, n$. We know $n \geq 1$ since $(e, e, \ldots, e)$ is fixed.

$\therefore \exists$ a f.p. $(a, a, \ldots, a)$, $a \neq e$.

$\Rightarrow a^p = 1$ + a has order p. //

Counting Orbits

G finite group.
X finite set.

$$F = \{(g,x) \in G \times X \mid gx = x\}$$

Def. $\mathrm{Fix}(g) = \{x \in X \mid gx = x\}$
"the fixed pt set of G".

Then let $I_F(g,x) = \begin{cases} 1 & \text{if } (g,x) \in F \\ 0 & \text{else} \end{cases}$.

Then $$\#F = \sum_{x \in X} \sum_{g \in G} I_F(g,x) = \sum_{x \in X} \#\mathrm{Stab}(x)$$

$$= \sum_{g \in G} \sum_{x \in X} I_F(g,x) = \sum_{g \in G} \#\mathrm{Fix}(g)$$

Thus $$\boxed{\sum_{x \in X} \#\mathrm{Stab}(x) = \sum_{g \in G} \#\mathrm{Fix}(g)}$$

and

$$\frac{1}{\#(G)} \sum_{g \in G} \# \mathrm{Fix}(g) = \sum_{x \in X} \frac{\# \mathrm{Stab}(x)}{\#(G)}$$

$$= \sum_{x \in X} \frac{1}{\# \mathcal{O}(x)}$$

$$= \sum_{\substack{\mathcal{O} \\ \text{all orbits}}} \sum_{x \in \mathcal{O}} \frac{1}{\# \mathcal{O}} = \sum_{\mathcal{O}} \frac{1}{\# \mathcal{O}} \sum_{x \in \mathcal{O}} 1$$

$$= \sum_{\mathcal{O}} 1 = \# \mathcal{O}$$

$$\therefore \boxed{\# \mathcal{O} = \frac{1}{\# G} \sum_{g \in G} \# \mathrm{Fix}(g)}$$

Burnside's Theorem.

The total number of orbits is equal to the average (over G) number of fixed pts of elements of G. //