First we list the homework problems due for week 6, Tuesday, February 25, 2014.

Please read the notes on permutations just after the problems. The further notes on matrices will be discussed in class.

Problems

I. page 10: 1.3.1, 1.3.2, 1.3.3
   page 15: 1.4.1
   page 23: 1.5.3, 1.5.4, 1.5.5, 1.5.6, 1.5.8
   page 73: 1.10.2, 1.10.3, 1.10.10

II. Let $C_3 = \{I, R, R^2 \mid R^3 = I\}$ be the cyclic group of order 3. Examine its multiplication table.

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>R</th>
<th>R^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>I</td>
<td>R</td>
<td>R^2</td>
</tr>
<tr>
<td>R</td>
<td>R</td>
<td>R^2</td>
<td>I</td>
</tr>
<tr>
<td>R^2</td>
<td>R^2</td>
<td>I</td>
<td>R</td>
</tr>
</tbody>
</table>

Note that each row of the table is a permutation of $I, R, R^2$.

Thus: $I \leftrightarrow (I, R, R^2)$
   $R \leftrightarrow (I, R^2, I)$
   $R^2 \leftrightarrow (R, I, R^2)$
In fact this must always be so.
For suppose you have a finite group \( G = \{ g_1, g_2, \ldots, g_n \} \) where this is a list of all the distinct elements of \( G \). Let \( g \in G \) be some given element of \( G \). Then
\[
 g \cdot g_i = g \cdot i_i
 g \cdot g_2 = g \cdot i_2
 \vdots
 g \cdot g_n = g \cdot i_n
\]
since \( G \) is closed under multiplication.

Claim. \( g \cdot i_k = g \cdot i_l \iff i = l \)

Proof. \( g \cdot i_k = g \cdot i_l \iff g \cdot g_k = g \cdot g_l \)
\[
 \iff g^{-1}(g \cdot g_k) = g^{-1}(g \cdot g_l)
 \iff (g^{-1}g)g_k = (g^{-1}g)g_l
 \iff I g_k = I g_l
 \iff g_k = g_l
 \iff k = l \text{ (since the list is a list of distinct elements).} \]

This means that \( \{ g \cdot i_1, g \cdot i_2, \ldots, g \cdot i_n \} \) is a permutation of \( \{ g_1, g_2, \ldots, g_n \} \).
This problem asks you to work out all the permutations associated with the six rows for the multiplication table for $S_3$. Where we take

$G = S_3 = \langle R, F \mid R^3 = I, F^2 = I, RF = FR^2 \rangle$

$= \{ I, R, R^2, F, FR, FR^2 \}$.

$g_1, g_2, g_3, g_4, g_5, g_6$

Here is an example of the calculation for one row: Let $g_4 = g_4 = F$

Then $g_4 g_1 = FI = F = g_4$

$g_4 g_2 = FR = g_5$

$g_4 g_3 = FR^2 = g_6$

$g_4 g_4 = FF = g_1$

$g_4 g_5 = FFR = R = g_2$

$g_4 g_6 = FFR^2 = R^2 = g_3$.

Let $[g_4] = (g_1 g_2 g_3 g_4 g_5 g_6)$

$[g_4] = [g_4] = (1 2 3 4 5 6)$

$[g_4] = [g_4] = (4 5 6 1 2 3)$. 
Work out the permutations for \([g_1], [g_2], [g_3], [g_4], [g_5], [g_6]\).

Since you know how to multiply (compose) permutations, you should find that
\[
[g_gh] = [g] \circ [h]
\]
when \(g\) and \(h\) are elements of the group. Try this out on your permutations. Can you write a proof of the formula \([gh] = [g] \circ [h]\)?

This exercise has you working and proving Cayley's Theorem that states that every finite group is isomorphic to a subgroup of the permutation group \(S_n\) when \(n\) is the number of elements of \(G\).
Redoing Diagrammatic Permutations

Regard \( S_n = \{ \sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\} \mid \sigma \text{ is 1-1 and onto} \} \)

Given \( \sigma \in S_n \), let \( \sigma(k) \) denote the image of \( k \in \{1, 2, \ldots, n\} \) under \( \sigma \). Given \( \rho, \sigma \in S_n \) define \( \rho \sigma = \rho \circ \sigma \); the composition of these functions. Thus \( (\rho \sigma)(k) = \rho(\sigma(k)) \) for \( k = 1, 2, \ldots, n \).

When we write \( \sigma = (1 \ 2 \ 3 \ 4) \in S_4 \) we mean: \( \sigma(1) = 2 \), \( \sigma(2) = 3 \), \( \sigma(3) = 4 \), \( \sigma(4) = 1 \).

If \( \rho = (1 \ 2 \ 3 \ 4) \): \( \rho(1) = 4 \), \( \rho(2) = 1 \), \( \rho(3) = 5 \), \( \rho(4) = 2 \).

\[ \begin{align*}
\rho \sigma(1) &= \rho(\sigma(1)) = 1 \\
\rho \sigma(2) &= \rho(\sigma(2)) = 3 \\
\rho \sigma(3) &= \rho(\sigma(3)) = 2 \\
\rho \sigma(4) &= \rho(\sigma(4)) = 4 \\
\end{align*} \]

You can do the multiplication of \( \rho \sigma \) by placing \( \rho \) next to \( \sigma \) and tracing what happens to each number.
In our diagramming we do it this way:

\[ \sigma (1 \ 2 \ 3 \ 4) = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix} \]

But note that

\[ \sigma \begin{pmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix} \]

We will use this ordering convention from now on.
I. More Matrices

\[
\begin{pmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{pmatrix}
= A = (a_{ij}), \text{ here } 3 \times 3.
\]

\[
\begin{pmatrix}
  a_{11} & \cdots & a_{1m} \\
  a_{21} & \cdots & a_{2m} \\
  \vdots & \ddots & \vdots \\
  a_{n1} & \cdots & a_{nm}
\end{pmatrix}
= A = (a_{ij}), \text{ n} \times \text{m}
\]

\[\text{n rows and m columns.}\]

\[\text{row}_i(A) = \text{vec}_i(A) = (a_{i1}, a_{i2}, \ldots, a_{im})\]

\[\text{col}_j(A) = \text{vec}_j(A) = \begin{pmatrix}
  a_{1j} \\
  a_{2j} \\
  \vdots \\
  a_{nj}
\end{pmatrix} \text{ jth col of } A.\]

Given \(A, n \times m\) and \(B, m \times p\)

we get \(C = AB, n \times p\) via

\[C_{ij} = \text{row}_i(A) \cdot \text{col}_j(B)\]

\[\text{dot product of vectors}\]
\[
C_{ij} = n \cdot (A) \cdot c_{ij}(B) \\
= (a_{i1}, \ldots, a_{im}) \cdot (b_{1j}, \ldots, b_{mj}) \\
= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}
\]

\[
(AB)_{ij} = \sum_{k=1}^{m} a_{ik}b_{kj}
\]

Sometimes we write

\[
(AB)_{ij} = \sum_{k=1}^{m} A_{ik}B_{kj}
\]

Diagram

\[
\begin{align*}
& A \\
\downarrow &
\\
e & A_j = A_{ij} \\
\end{align*}
\]

Product

\[
\begin{align*}
& A \quad B \\
\rightarrow &
\\
e & A_kB = \sum_{i,j} A_{ik}B_{kj}
\end{align*}
\]
Examples: \((a \ b \ c)(d) = ad + be + cf\),
\[
\begin{pmatrix}
(a & b & c) \\
(c & d & e)
\end{pmatrix}
\begin{pmatrix}
f \\
g \\
h
\end{pmatrix}
= (ae + bf)
\begin{pmatrix}
ce + df
\end{pmatrix}
\]

\(\begin{pmatrix}
a & b & c \\
d & e & f
\end{pmatrix}
\begin{pmatrix}
1 & 4 & 7 \\
2 & 5 & 8
\end{pmatrix}
= \begin{pmatrix}
(a+2b+3c) & 4e+5b+6c & 7c+8b+9c \\
d+2e+3f & 4d+5e+6f & 7d+8e+9f
\end{pmatrix}
\]

\[3 \times 3, \ 3 \times 3\]
\[
\begin{pmatrix}
d & a & d & b & d & c \\
ea & e & b & e & c
\end{pmatrix}
= \begin{pmatrix}
fa & fb & fc
\end{pmatrix}
\]

\[3 \times 1, \ 1 \times 3 \quad 3 \times 3\]

**Notation:** \(\vec{a} = (a_1 \ a_2 \ \cdots \ a_n)\), \(\vec{a}^T = (a_1, a_2, \ldots, a_n)\)

Thus, \(\vec{a}^T \vec{b} = (a_1 \cdots a_n) / (b_1 \cdots b_n) = a_1 b_1 + \cdots + a_n b_n = \|\vec{a} \cdot \vec{b}\|\)

But \(\vec{a} \vec{b}^T = \begin{pmatrix}
a_1 \\
a_2 \\
\vdots \\
a_n
\end{pmatrix}
\begin{pmatrix}
b_1 & b_2 & \cdots & b_n
\end{pmatrix}
= \begin{pmatrix}
a_1 b_1 & a_1 b_2 & \cdots & a_1 b_n \\
a_2 b_1 & a_2 b_2 & \cdots & a_2 b_n \\
\vdots & \vdots & \ddots & \vdots \\
a_n b_1 & a_n b_2 & \cdots & a_n b_n
\end{pmatrix}
\]
is a matrix.
**Dirac Notation**

\[ |\vec{a}\rangle = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \]

\[ \langle \vec{a} | = (a_1, a_2, \ldots, a_n) \]

Thus \( \langle \vec{a} | \vec{b} \rangle = \langle \vec{a} | b \rangle = \vec{a} \cdot \vec{b} \)

But \( |\vec{a}\rangle \langle \vec{b}| = \vec{a} \vec{b}^T \) is an \( n \times n \) matrix.

---

**Permutation Matrices**

\[ (1, 0) = I, \quad (0, 1) = T \]

Note \( I(a, b) = (a, b), \quad T(a, b) = (0, 1)(a, b) = (b, a) \).

\[ (1, 0, 0) = I, \quad (0, 1, 0) = R, \quad (0, 0, 1) = F \]

\[ R(a, b, c) = (0, 1, 0)(a, b, c) = (b, c, a) \]

\[ F(a, b, c) = (1, 0, 0)(a, b, c) = (a, c, d) \]

\[ \text{You see that by permuting rows of } I \]

\[ \text{you can obtain all } 6 \text{ permutations and get a set of } 6 \text{ matrices that represent } S_3. \]
Determinants

(a) Signs of permutations

\[ \text{sgn}(\pi) \overset{\text{def}}{=} \left\lfloor \begin{array}{c}
\# \text{ of transpositions} \\
(i \ldots j) \mapsto (j \ldots i)
\end{array} \right\rfloor \]

needed to take the permutation back to standard order.

\[ (-1)^2 \]

e.g.

\[ 3 \overset{1}{\rightarrow} 13 \overset{2}{\rightarrow} 123 \]

\[ \Rightarrow \text{sgn}(312) = +1 = (-1)^2 \]

\[ (1 \ 2 \ 3) \overset{1}{\leftrightarrow} (2 \ 3 \ 1) ; \]

\[ (3 \ 1 \ 2) \overset{2}{\leftrightarrow} (2 \ 3 \ 1) \]

2 transpositions

or

Here we indicated the actual switching.

Here we factorized the mapping into transpositions.
\[
\begin{align*}
(1 & 2 & 3) = R \\
\text{sgn}(R) = (-1)^3 = -1.
\end{align*}
\]

You can see that if you represent a permutation in \( S_n \) by a line diagram

\[
\text{Then } \# \text{(crossings)} \\
\text{sgn}(\sigma) = (-1).
\]

\[
\{ III, XXX, XXX, XXX \} = S_3
\]

Note that half the elements of \( S_3 \) have + signs and half have - signs.

**Theorem.** \( \text{sgn} : S_n \rightarrow \{+1, -1\} = C_2 \)

Then \( \text{sgn}(1) = 1 \) and for \( \sigma, \tau \in S_n \), \( \text{sgn}(\sigma \tau) = \text{sgn}(\sigma) \text{sgn}(\tau) \).
Proof. Represent \( \sigma \) and \( \tau \) by string diagrams so that

\[
\text{sgn}(\sigma) = (-1)^{c(\sigma)}, \quad \text{sgn}(\tau) = (-1)^{c(\tau)}
\]

where \( c(\sigma) \) = \# of crossings in the \( \sigma \)-diagram and \( c(\tau) \) = \# crossings in the \( \tau \)-diagram.

Then clearly \( c\left[ \begin{array}{c} \sigma \\ \tau \end{array} \right] = c(\sigma) + c(\tau) \).

\[
\therefore \quad \text{sgn}(\sigma \tau) = (-1)^{c(\sigma) + c(\tau)}
\]

\[
= (-1)^{c(\sigma)} (-1)^{c(\tau)}
\]

\[
= \text{sgn}(\sigma) \text{sgn}(\tau)
\]

Example. \( \text{sgn}(XX) = (-1)^2 = +1 \)

\( \text{sgn}(X|) = (-1)^1 = -1 \)

\( \text{sgn}(X|) = (-1)^2 + 1(-1)^3 = -1 \)

\( \text{sgn}(\text{in}) = -1 \)
Let \( A_\infty = \{ \sigma \in S_n \mid \sigma(0) = +1 \} \).
You can easily check that \( A_n \) is a subgroup of \( S_n \).
For example,
\[
A_3 = \{ \begin{array}{lll}1 & 2 & 3 \\ 3 & 1 & 2 \end{array} \} < S_3.
\]
In general \( A_n \) will have \( (n!) / 2 \) elements. Thus \( A_4 \) should have \( (4!) / 2 = 24 / 2 = 12 \) elements.

Make a list of the elements of \( A_4 \) and experiment with their products.

(b) **Permutations and Determinants**

You can regard this pattern as the permutation
\[
(1, 2, 3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}
\]
where the dot is in column \( i \) and row \( i \).
Think of this as

\[
\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{array}
\] \leftrightarrow
\begin{pmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
\end{pmatrix}

and

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
\end{bmatrix}
\begin{pmatrix}
1 \\
2 \\
3 \\
\end{pmatrix}
=
\begin{pmatrix}
2 \\
3 \\
1 \\
\end{pmatrix}
\]

Thus you can find the corresponding permutation by matrix multiplication.

Work out the corresponding permutation and matrix multiplcation.

\[
\text{Now Note: } \det\begin{vmatrix}
a & b \\
c & d \\
\end{vmatrix} = ad - bc
\]

We will use this pattern to generalize to $\det(A)$ when $A$ is $n \times n$.

\[
\begin{array}{ccc}
a & d \\
\end{array}
\begin{array}{ccc}
c & b \\
\end{array}
\]

\[
\begin{array}{ccc}
\cdot & \cdot \\
\cdot & \cdot \\
\end{array}
\begin{array}{ccc}
\cdot & \cdot \\
\cdot & \cdot \\
\end{array}
\]

$\text{sgn} = +1$

$\text{sgn} = -1$
On an nxn grid, a permutation pattern \( P \) is a placement of dots in the grid so that each dot occupies a unique row and a unique column.

As we have seen, there is a 1-1 correspondence between permutation patterns \( P \) on an nxn grid and elements of \( S_n \).

\[
\begin{array}{c}
\text{Let } P_n = \{ P \mid P \text{ is a permutation pattern on an nxn grid} \}
\end{array}
\]

\[
\begin{array}{c}
\text{Let } \sigma : P_n \rightarrow S_n \\
\sigma (P) = \text{elt of } S_n \text{ that corresponds to } P.
\end{array}
\]

Define \( \text{sgn}(P) = \text{sgn}(\sigma (P)) \).
Thus \( \text{sgn} \begin{pmatrix} a & \ast \\ \ast & d \end{pmatrix} = +1 \)
\[ \text{sgn} \begin{pmatrix} b & \ast \\ \ast & c \end{pmatrix} = -1. \]

Define for an \( n \times n \) matrix \( A \)
\[ \langle A | P \rangle = \text{product of the entries of } A \text{ that correspond to dots in } P. \]

\[ \langle (a, b) | \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \rangle = ad \]
\[ \langle (c, d) | \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \rangle = bc. \]

Now we can define the determinant of an \( n \times n \) matrix \( A \):
\[ \text{Det}(A) = \sum_{P \in \Pi_n} \text{sgn}(P) \langle A | P \rangle. \]
\[ \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = aeK + bfg + cdh - ceq - bdk - afh. \]

You can verify that:

1. \( \det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) A_{1 \sigma(1)} A_{2 \sigma(2)} \cdots A_{n \sigma(n)}. \)

2. Regard \( D_{\text{et}}(A) = D \left( \begin{array}{c} r_1 \\ r_2 \\ \vdots \\ r_n \end{array} \right) \) where \( r_i = i \text{th row of } A. \)

- Then \( D(A') = -D(A) \) if \( A' \) obtained from \( A \) by interchanging two rows.

- \( D \left( a r_i + b r_i' \right) = a D(r_i) + b D(r_i') \).

That is, \( D \) is a linear function of each row.
One can show that an n×n matrix $A$ is invertible ($\exists B$ s.t. $AB = I$) if and only if $\text{Det}(A) \neq 0$. In fact there is an explicit formula for $A^{-1}$ obtained as follows: Given $A$ an $n \times n$ matrix, define the $i,j$ cofactor $C_{ij}$ by the formula

$$C_{ij} = (-1)^{i+j} \text{Det}(A[i,j])$$

where $A[i,j]$ is the $(n-1) \times (n-1)$ matrix obtained from $A$ by removing row $i$ and column $j$ from $A$.

E.g., $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ then $C_{12} = \text{Det} \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix}$

$$\underline{= (-1)^{1+2} \text{Det}(4,6)} = - \text{Det}(4,6) = -(36 - 42) = -(-6) = 6.$$  

Let $C = (C_{ij})$. This is the cofactor matrix for $A$.

Then we have the fact that $C^t A = \text{Det}(A) I$ where $(C^t)_{ij} = C_{ji}$ is the transpose of $C$, and $I$ is the $n \times n$ identity matrix.
e.g. \[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]

\[ C = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}, \quad C^T = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \]

\[ C^T A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]

\[ = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} \]

\[ = (ad-bc) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ C^T A = \text{Det}(A) I \]

From this it follows that, when \( \text{Det}(A) \neq 0 \), then

\[ A^{-1} = \frac{1}{\text{Det}(A)} C^T. \]

Exercise: Assuming \( A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \) has \( \text{Det}(A) \neq 0 \). Find the formula for \( A^{-1} \).