Euler Formula Facts

1. \( G \subset \mathbb{R}^2 \), \( G \) a connected graph
   \[ v - e + f = 2. \]

2. \( G \subset S_g \), \( S_g \) an orientable surface
   of genus \( g \), \( G \) a tight embedding
   (all regions are disks after removal of the
   boundary of the region), \( G \) connected.
   \[ v - e + f = 2 - 2g. \]

3. \( G \subset \mathbb{P}^2 \) (\( \mathbb{P}^2 \) = the projective plane)
   \( G \) a tight graph embedding, \( G \) connected.
   \[ v - e + f = 1. \]

4. \( G \subset S_g, \mathbb{R} = \) sphere with \( g \) handles
   and "\( r \) cross-caps."
   \[ = T \# T \# \cdots \# T \# P \# P \# \cdots \# P \]
   \( G \) tight, connected.
   \[ v - e + f = 2 - 2g - r. \]
Note that \( 4^0 \) is compatible with the fact that
\[
P \# P \# P \cong \text{Klein} \# P \cong \text{T} \# P
\]

since \( 2 - 2 \cdot 1 - 1 = 2 - 3 \cdot 1 \).

Since \( P \cong \text{Mobius} \cup \text{Disk} \), taking connected sum with \( P \) is same as cutting a single hole in the surface of then gluing a Mobius strip to the boundary of the hole.

This operation is called "adding a cross-cap." Sometimes you will see a singular picture of the Mobius strip:

This singular Mobius with circular boundary is also called a cross-cap.
\[ S_{2,1} = \text{sphere with two handles and one cross-cap.} \]

Exercise: \[ S' = S_{n,m} \]
Find \[ n = \quad \]
\[ m = \quad \]
Penrose Formula

\[ \sum_{\alpha, \beta, \gamma \in \{1, 2, 3\}} (V^2) E_{abc} \]

where

\[ \begin{cases} 
E_{123} = 1, & E_{213} = -1 \\
E_{312} = 1, & E_{321} = -1 \\
E_{231} = 1, & E_{132} = -1 
\end{cases} \]

\( G \) a trivalent plane graph.

Define \[ [G] = \sum_{\alpha, \beta, \gamma, \delta \in \{1, 2, 3\}} (V^2) E_{abcd} \]

we take one epsilon tensor for each vertex of \( G \), and label all the edges of the graph differently.

e.g. \[ [G] = [G] = \sum_{\alpha, \beta, \gamma, \delta \in \{1, 2, 3\}} (V^2) E_{abcd} (V^2) E_{bed} \]

\[ = \sum_{\alpha, \beta, \gamma, \delta \in \{1, 2, 3\}} (V^2) E_{abcd} (V^2) E_{bed} (V^2) E_{cedf} \]

Thm (a) \[ [G] = \text{# of edge 3-colorings of } G \text{ (3 distinct colors per vertex)} \]
when \( G \subset \mathbb{R}^2 \) planar.

(b) \[ [X] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - [X] \]

where \[ [G] = 3 \begin{bmatrix} 6 \end{bmatrix} \]

\[ [G] = 3. \]

(c) \[ [Y] = - [Y] \]
(b) comes from the "epsilon identity"

\[ \sum_{c} \epsilon_{abc} \epsilon_{cde} = -\delta_{ed} \delta_{dc} + \delta_{d} \delta_{ce} \]

\[ \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \]

\[ \epsilon_{ijk} = \begin{cases} 1 & i = 1, j = 2, k = 3 \\ -1 & i = 1, j = 3, k = 2 \\ 0 & \text{otherwise} \end{cases} \]

\[ \epsilon_{123} \epsilon_{132} = (-1)(-1) = 1 \]

\[ \epsilon_{123} \epsilon_{213} = (-1)^{1} (-1)^{2} = (-1) \]

\[ \epsilon_{213} = \epsilon_{123} = +1 \]

\[ \epsilon_{132} = \epsilon_{213} = -1 \]

(a) comes from

\[ e \equiv 2 \]

\[ r \equiv 1 \]

\[ \sum \epsilon_{rpb} = \epsilon_{123} = +1 \]

\[ \sum \epsilon_{brp} = \epsilon_{213} = -1 \]
\[ (+\sqrt{i}) (-\sqrt{i}) = +1 \]

each bounce contributes +1

\[ (+\sqrt{-1}) (+\sqrt{-1}) = -1. \]

\[ (-\sqrt{-1}) (-\sqrt{-1}) = -1. \]

each crossing contributes -1.

\[ \therefore \text{each } \prod (\sqrt{-1} \varepsilon_{abc}) \text{ for a given coloring contributes } (-1)^{\# \text{crossings}}. \]

But \# crossings is even (by Jordan curve theorem) \[ \therefore \text{each product contributes } +1 \text{ and so } \]

\[ \# \text{crossings counts colorings.} \]

\[ \text{e.g.} \]

\[ \begin{array}{c}
\text{contributes } (-1)^2 = +1.
\end{array} \]

\[ (\varepsilon) \text{ is implied by } \varepsilon_{abc} = -\varepsilon_{bac}. \]