COMPLEX NUMBERS AND POLYNOMIAL EQUATIONS

by

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I. Solving Quadratic Equations by Iteration.

You already know that $ax^2 + bx + c = 0$ has the general solution $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Thus, for example, $x^2 - x - 1 = 0$ has the roots $x = \frac{1 + \sqrt{5}}{2}$ and $x = \frac{1 - \sqrt{5}}{2}$. Let's look at this in another way. If

$$x^2 - x - 1 = 0$$

then
$$x^2 = x + 1$$

and therefore
$$x^2/x = (x+1)/x$$

so
$$x = 1 + 1/x.$$

Thus if we define the function $f(x) = 1 + 1/x$, then a solution to our quadratic equation will be a <u>fixed point</u> x such that $x = f(x)$. It will be a point left fixed by the function f.

The reason that this is interesting is that we can actually approximate a solution by <u>iterating</u> f. Try the following experiment on a calculator:

$$\underline{f(x) = 1 + 1/x}$$

$$f(1) = 1 + 1/1 = 2$$
$$f(2) = 1 + 1/2 = 1.5$$
$$f(1.5) = 1.666\cdots$$
$$f(1.666\cdots) = 1.625$$
$$f(1.625) = 1.615\cdots$$
$$f(1.615\cdots) = 1.619\cdots$$
$$f(1.619\cdots) = 1.617\cdots$$
$$f(1.617\cdots) = 1.618\cdots \approx \frac{1 + \sqrt{5}}{2}$$

As we keep applying f over and over again, we get closer and closer to the root $(1 + \sqrt{5})/2$.

This provides a way of numerically approximating the roots of an equation. Hence it can be used to advantage if you have a calculator or a small computer. The method can be used for other equations than degree 2. You might like to try

$$x^3 - x^2 - x - 1 = 0$$
$$x^3 = x^2 + x + 1$$
$$x^3/x^2 = (x^2 + x + 1)/x^2$$
$$x = 1 + (1/x) + (1/x^2)$$

So let $g(x) = 1 + \frac{1}{x} + \frac{1}{x^2}$ and try $g(1)$, $gg(1)$, $ggg(1), \ldots$

$$\boxed{\text{Let } f^n(x) = \overbrace{fff\ \ldots\ f}^{n \text{ times}}(x)}$$

Getting back to $f(x) = 1 + 1/x$, note that

$$f(1) = 1 + 1/1$$

$$f^2(1) = 1 + \frac{1}{f(1)} = 1 + \frac{1}{1 + \frac{1}{1}}$$

$$f^3(1) = 1 + \frac{1}{f^2(1)} = 1 + \frac{}{1 + \frac{1}{1 + \frac{1}{1}}}$$

<u>Exercise 1</u>. Using f as above, show by direct computation that

$$\begin{aligned} f(1) &= 2/1 \\ f^2(1) &= 3/2 \\ f^3(1) &= 5/3 \\ f^4(1) &= 8/5 \\ f^5(1) &= 13/8 \end{aligned}$$

and generally that $f^{n+1}(1)$ is obtained by

1) add numerator and denominator of $f^n(1)$.
   This will be the new numerator of $f^{n+1}(1)$.

2) The denominator of $f^{n+1}(1)$ is the numerator of $f^n(1)$.

Thus

$$f^6(1) = (13+8)/13 = 21/13$$

$$f^7(1) = (21+13)/21 = 34/21$$

etc...

The series of integers

$$\begin{array}{cccccccccccc} 1, & 1, & 2, & 3, & 5, & 8, & 13, & 21, & 34, & 55, & 89, & \ldots \\ & & \| & \| & \| & \| & \| & \| & & & & \\ & & 1 & 2 & 3 & 5 & 8 & 13 & \ldots & & & \\ & & + & + & + & + & + & + & & & & \\ & & 1 & 1 & 2 & 3 & 5 & 8 & & & & \end{array}$$

is called the <u>Fibonacci</u> <u>Series</u> (invented by an Italian mathematician - Fibonacci in the 1500's) and has many interesting properties. For example

$$1\cdot 2 = 1^2 + 1$$
$$1\cdot 3 = 2^2 - 1$$
$$2\cdot 5 = 3^2 + 1$$
$$3\cdot 8 = 5^2 - 1$$
$$\cdots$$

In particular, we have seen that the fractions $\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \ldots$ converge to $\frac{1+\sqrt{5}}{2}$.

<u>Exercise 2</u>. Let's look at $x^2 - 2x - 1 = 0$.

a) Show that $x^2 - 2x - 1 = 0$ has roots $x = 1 \pm \sqrt{2}$.

b) Show that x is a solution of $x^2 - 2x - 1$ if and only if $h(x) = x$ where $h(x) = 2 + 1/x$.

c) Compute $h(1)$, $h^2(1)$, $h^3(1)$, $h^4(1)$ using a calculator. Compare $h^4(1)$ with $1 + \sqrt{2} = 1 + 1.414\cdots = 2.414\cdots$

d) By working out the fractions

$$h(1) = 2 + 1/1 = 3$$
$$h(3) = 2 + 1/3 = 7/3$$
$$h(7/3) = 2 + 3/7 = 17/7$$
$$h(17/7) = 2 + \frac{7}{17} = \frac{41}{17}, \quad h\left(\frac{41}{17}\right) = \frac{99}{41}$$

we get a series of numbers

$$3, 7, 17, 41, 99, \ldots$$

What are the next few numbers in this series?

What does the series have to do with $\sqrt{2}$?

HINT: Compare $99/41$ with $1 + \sqrt{2}$.

e) Explain why the equation

$$2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}} = 1 + \sqrt{2}$$

makes sense!

HINT: From what we have done you should begin to see that $\frac{1 + \sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$ makes sense in terms of the approximations. Also, if $x = 1 + \cfrac{1}{\left\{1 + \cfrac{1}{1 + \cdots}\right\}}$, then $x = 1 + \frac{1}{x}$.

f) What number is represented by

$$3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cdots}}}}}} \; ?$$

HINT: If $x = 3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cdots}}}$

Then $x = 3 + \frac{1}{x}$.

g) What number is represented by

$$\sqrt{2} + \cfrac{1}{\sqrt{2} + \cfrac{1}{\sqrt{2} + \cfrac{1}{\sqrt{2} + \cfrac{1}{\sqrt{2} + \cdots}}}} \quad ?$$

Exercise 3. What number is represented by

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}} = w?$$

(HINT: $w = \sqrt{1 + w}$)

II. But Not All Quadratics Solve by Iteration.

The iteration method doesn't always work! You know that some quadratic equations don't even have any real roots. For example $x^2 + 1 = 0$. What happens if we try iteration here? Well

$$x^2 + 1 = 0$$
$$x^2 = -1$$
$$x^2/x = -1/x$$
$$x = -1/x.$$

Let

$$F(x) = -1/x$$
$$F(1) = -1$$
$$FF(1) = F(-1) = +1$$
$$FFF(1) = F(+1) = -1$$
$$FFFF(1) = F(-1) = +1$$
$$\ldots$$

And so it goes, just oscillating back and forth between $+1$ and -1.

In order to get a solution to $x^2 + 1 = 0$ mathematicians had to invent a <u>new number</u> $i = \sqrt{-1}$ so that $i^2 = -1$. You already known quite a lot about this.

It is, however, very interesting to see what kind of patterns arise when we attempt to solve a quadratic, whose roots are imaginary, by the iteration process.

Example: $x^2 - x + 1 = 0$ so $x = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}\,i}{2}$.

Iteration: $x^2 = x - 1$

$x = 1 - 1/x$

Let $F(x) = 1 - 1/x$

$F(2) = 1 - 1/2 = 1/2$

$F^2(2) = 1 - 2 = -1$

$F^3(2) = 1 - \left(\frac{1}{-1}\right) = 2$

$F^4(2) = 1 - 1/2 = 1/2.$

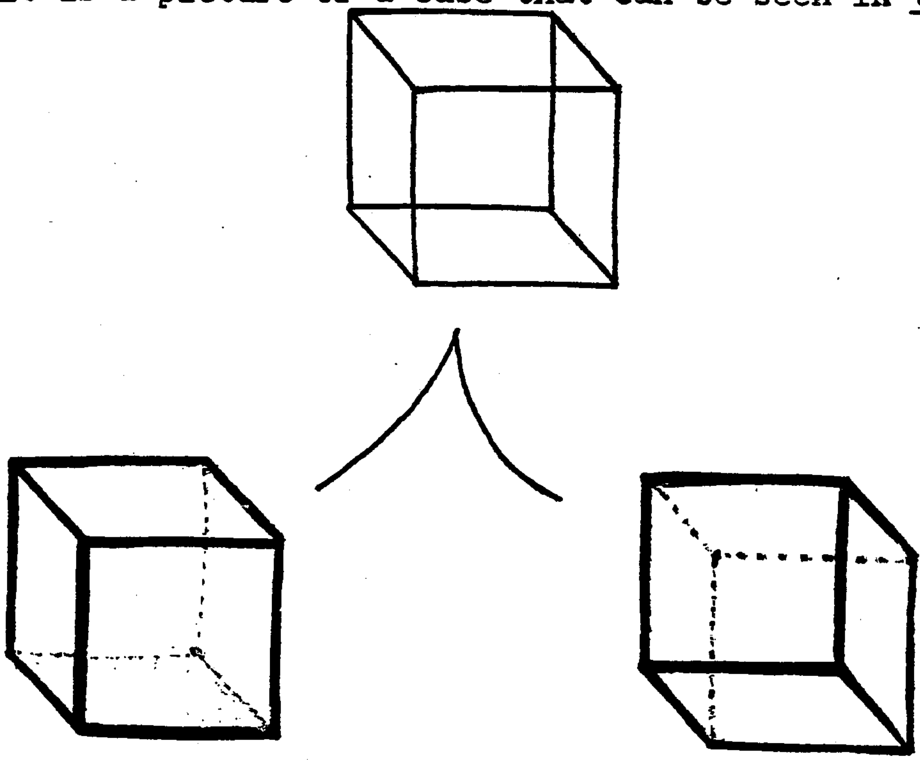
So $F^4(2) = F^1(2)$ and the sequence has

period 3: $\frac{1}{2}, -1, 2, \frac{1}{2}, -1, 2, \frac{1}{2}, -1, 2, \ldots$

Extra Credit Exercise 4: Try out more examples of this type. For example, investigate $x^2 - x + 2 = 0$ along these lines. That is, try iterating $f(x) = 1 - 2/x$.

III. Complex Numbers and Optical Illusions.

You are probably familiar with the so-called Necker cube illusion: It is a picture of a cube that can be seen in two ways.



The illusion, and the way we tend to oscillate back and forth between the two views, is generated by the ambiguity of  which the mind interprets as a crossing of type  or of type . That is, like  or  .

Of course both cubic views are just in your mind, and certainly the difference between the two views is purely imaginary (a matter of image-ination)!

I like to think of $+\sqrt{-1}$ and $-\sqrt{-1}$ as being something like the two views of the Necker cube. They are both our attempts to "jump out" of the oscillating paradox presented by $x = -1/x$. How, you may ask, do you arrive at two views from the paradox?

Well, let's see (I'd better think fast.), the paradox arises by trying to solve $x = -1/x$ by iteration. We start with -1 and get $+1$, then -1, then $+1$,...

$$-1, +1, -1, +1, -1, +1, -1, +1, -1, +1, \ldots$$

But we could have started with $+1$ and then

$$+1, -1, +1, -1, +1, -1, +1, -1, +1, -1, \ldots$$

If you and your friend had each started at the same time, one with $+1$, the other with -1, you'd find yourselves chanting opposite numbers at the same time! Each taking a different view of $\sqrt{-1}$.

Let's put this another way. Suppose I start intoning:

PLUS, MINUS, PLUS, MINUS, PLUS, MINUS,...

and you listen to me. After a while it will start sounding <u>PLUS</u> MINUS, <u>PLUS</u> MINUS,... and then it will switch to <u>MINUS</u> PLUS, <u>MINUS</u> PLUS, <u>MINUS</u> PLUS,... <u>I'm</u> not doing that! You're doing it. The difference is in your imagination, and you get two views (soundings really) of my chanting in a way perfectly analogous to the Necker cube illusion!

Thus we might try to say that $\sqrt{-1}$ is "really" the pair [+1,-1] (or is it [-1,+1]?) where this denotes the PLUS MINUS take on the sequence ... PLUS MINUS PLUS MINUS...

It is actually possible to make mathematics out of these ideas. The algebraic version is called matrix algebra. In matrix algebra we represent $\sqrt{-1}$ by an array $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $-\sqrt{-1}$ by an array $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. (I'll tell you about the zeroes in a moment). Notice how $\boxed{\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}}$ and $\boxed{\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}}$ are the basic ingredients in the array, just as we have wanted.

The zeroes are locations for the real part (which does not oscillate). In general a complex number $a + b\sqrt{-1}$ is represented by the array $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. Thus $3 + 4\sqrt{-1}$ corresponds to $\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$.

In this system the array $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ corresponds to $1 + 0\sqrt{-1} = 1$, the number one. The next exercise shows you how to multiply these matrices to make things work out right. For more information read the chapter on matrices in the text.

Optional Exercise 5. Define the product of two matrices by the formula

$$\begin{bmatrix} x & z \\ w & y \end{bmatrix}\begin{bmatrix} A & C \\ D & B \end{bmatrix} = \begin{bmatrix} xA + zD & xC + zB \\ wA + yD & wC + yB \end{bmatrix}.$$

Thus

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1\cdot1 + 2\cdot2 & 1\cdot1 + 2\cdot1 \\ 3\cdot1 + 4\cdot2 & 3\cdot1 + 4\cdot1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 11 & 7 \end{bmatrix}$$

<u>Show</u>: a) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} A & C \\ D & B \end{bmatrix} = \begin{bmatrix} -A & -C \\ -D & -B \end{bmatrix}.$

b) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$

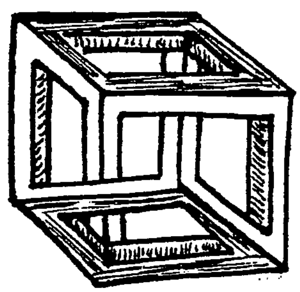
(This corresponds to $(\sqrt{-1})^2 = -1$.)

c) $\begin{bmatrix} a & b \\ -b & c \end{bmatrix}\begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} ac-bd & ad+bc \\ ad+bc & ac-bd \end{bmatrix}.$

This corresponds to $(a+bi)(c+di) =$
$(ad-bc)+(ad+bc)i.$

<u>Exercise 6</u>. Give as many examples as you can of ambiguous situations (like the Necker cube) that have multiple interpretations. (Look at other optical illusions, word games, things you see and hear,... perhaps after awhile and some imagining you'll begin to wonder what doesn't have a multiple interpretation!)

<u>Exercise 7</u>. <u>Discuss</u>:

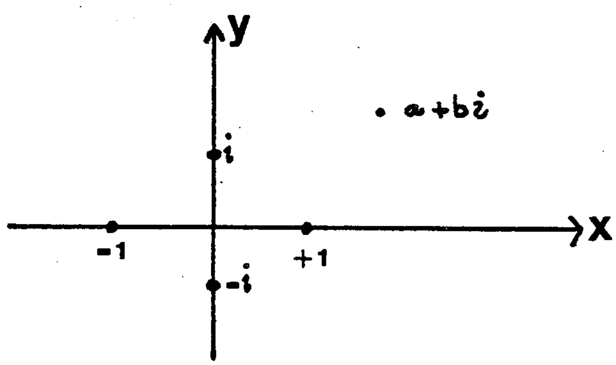


<u>Exercise 8</u>. <u>Discuss</u>:

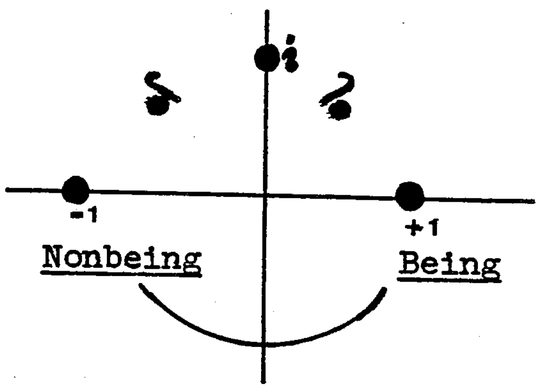
$$-1 = (\sqrt{-1})^2 = (\sqrt{-1})(\sqrt{-1})$$
$$= \sqrt{(-1)(-1)}$$
$$= \sqrt{+1}$$
$$\therefore \quad -1 = +1$$

IV. The Geometric Interpretation of $\sqrt{-1}$.

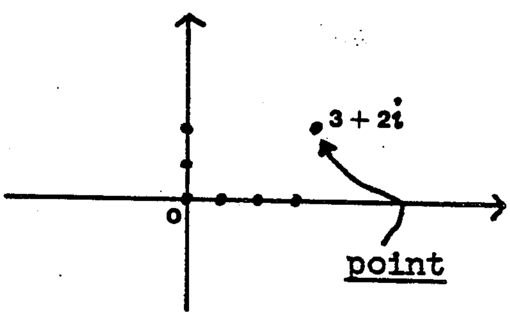
The paradoxical nature of $\sqrt{-1}$ gave early mathematicians much uneasiness (just as at an even earlier time there was some anxiety over irrationals like the $\sqrt{2}$ and π) until Carl Friedrich Gauss, in the early 1800's, gave a beautiful geometric interpretation. (Also due to C. Wessel, a Norwegian surveyor and J.R. Argand, a Swiss self-taught bookkeeper.) The geometric interpretation is simplicity itself! We let $a+bi$ $(i=\sqrt{-1})$ correspond to the point with coordinates (a,b) in the Cartesian plane. Thus i resides on the y-axis and stays out of the affairs of reality, which are conducted along the x-axis. That is, the purely real numbers lie on the x-axis.

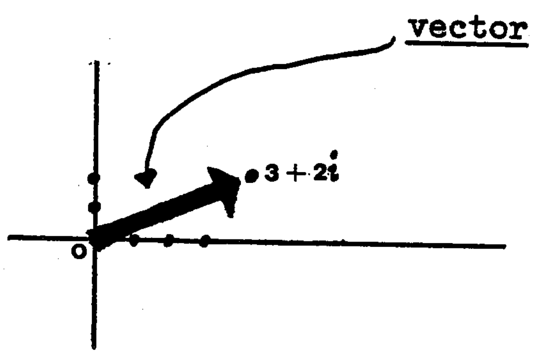


This diagram is a remarkable geometric embodiment of the statement of Leibniz (1700's): "The Divine Spirit found a sublime outlet in that wonder of analysis, that portent of the ideal world, that amphibian between being and non-being, which we call the imaginary root of negative unity."

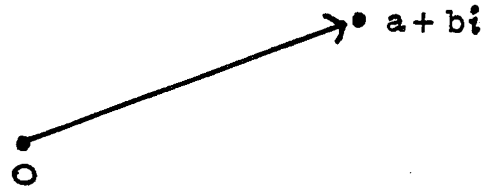


We shall often represent a point by the <u>vector</u> from the origin to that point. <u>Thus</u>

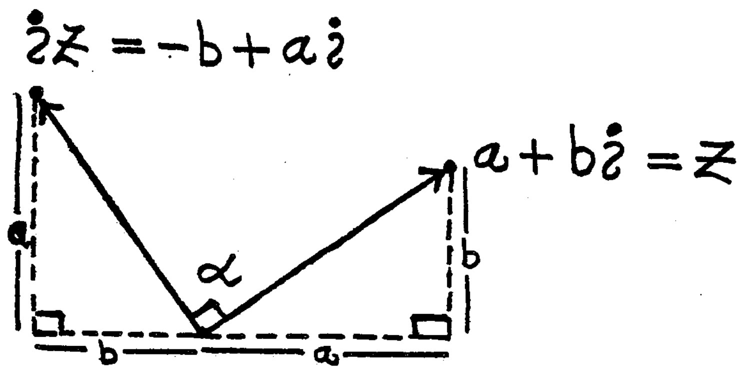




The <u>vector corresponding to a point</u> is just a directed line segment from the origin to that point. We usually draw it as an arrow, with <u>the tip of the arrow touching the point</u>.



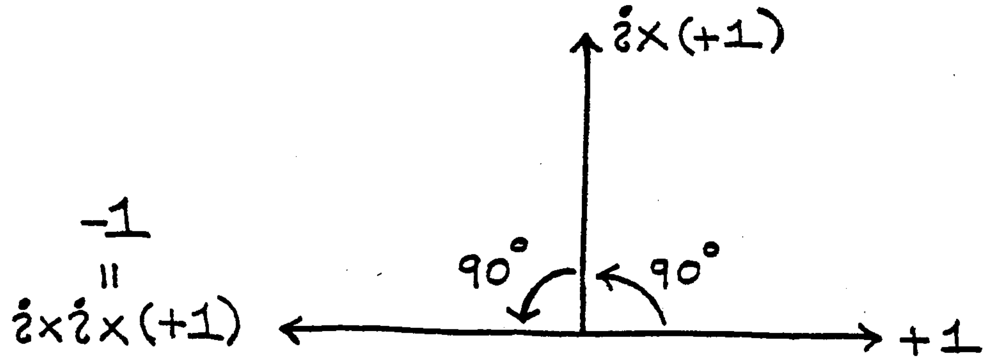
The first fact that dawns with this geometry is that <u>multiplication by</u> $i = \sqrt{-1}$ <u>rotates vectors by</u> 90°.

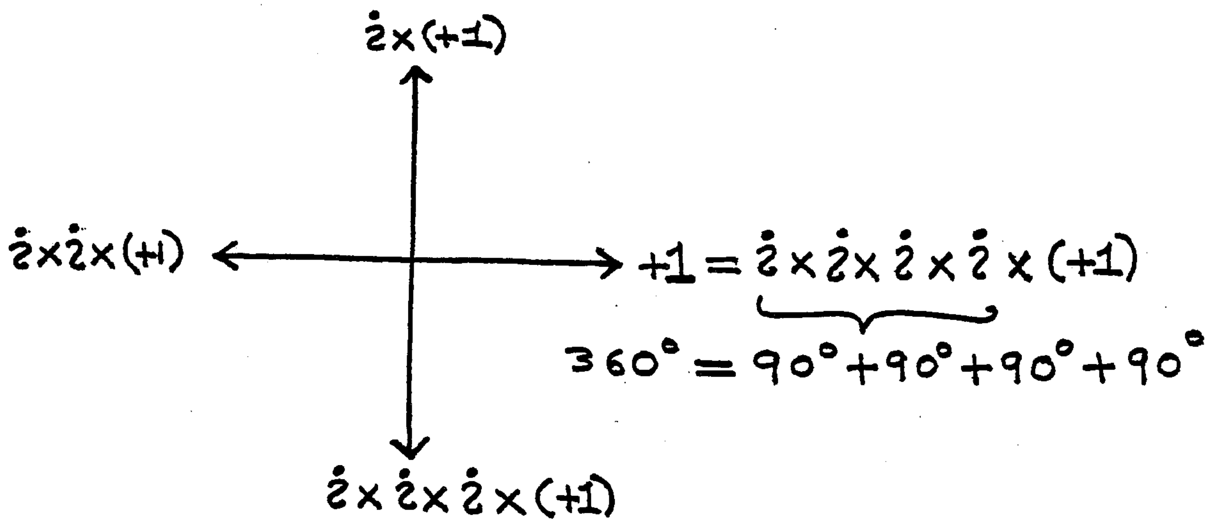


<u>Exercise 9</u>. Using the geometry in the above diagram, explain why angle α is necessarily 90°.

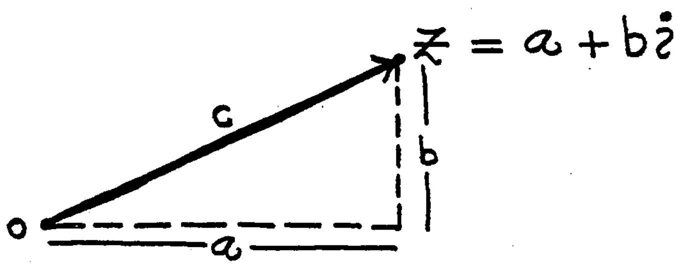
<u>Exercise 10</u>. Plot $3 + 4i$ and $i(3+4i) = 3i - 4 = -4+3i$ on graph paper. Show graphically that the vectors for $3+4i$ and $-4+3i$ make a 90° angle.

Note that we now can understand geometrically why $ii = -1$:





There is much more to this. First of all the lengths of the vectors are contained intrinsically in the algebra of the complex numbers. View the next diagram:



$$\underline{\text{Pythagoras}} \Longrightarrow c^2 = a^2 + b^2$$

Thus $c = \sqrt{a^2 + b^2}$ is the length of the vector corresponding to $a + bi$. Now recall that if $z = a + bi$ and $\bar{z} = a - bi$ (its complex conjugate) then

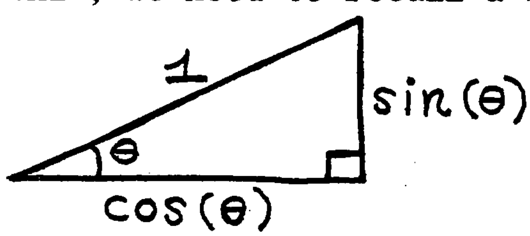
$$\begin{aligned} z\bar{z} &= (a+bi)(a-bi) \\ &= (a)^2 - (bi)^2 \\ &= a^2 + b^2. \end{aligned}$$

Thus $c = \sqrt{z\bar{z}}$, and lengths can be defined entirely in the algebra.

<u>Exercise 11</u>. a) Let $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. Compute $z\bar{z}$.

b) Let $z = a + bi$, $w = c + di$. Using $\overline{zw} = \bar{z}\bar{w}$, show that $(z\bar{z})(w\bar{w}) = (zw)(\overline{zw})$ and hence, $(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2$. (<u>HINT</u>: $zw = (ac-bd) + i(ad+bc)$).

Furthermore, it is not just multiplication by $i = \sqrt{-1}$ that has a geometric interpretation. <u>There is, in fact, a simple interpretation of the product of any two complex numbers</u>. In order to understand this, we need to recall a bit of <u>trigonometry</u>:

a) 

definition of $\sin(\theta)$ and $\cos(\theta)$.

$$\Longrightarrow \quad \sin^2(\theta) + \cos^2(\theta) = 1$$

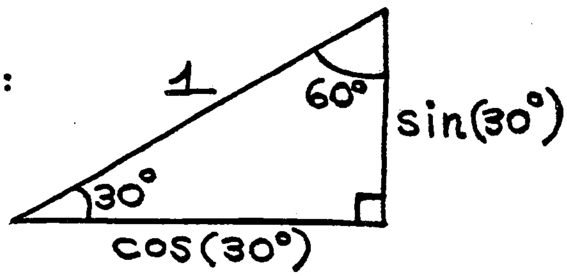
$$(\sin^2(\theta) = (\sin(\theta))^2)$$

b) $\sin(\theta + \phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)$

$\cos(\theta + \phi) = \cos(\theta)\sin(\phi) - \sin(\theta)\sin(\phi)$

These are the angle-addition formulas. Please, just be willing to use them. We won't prove them, but if you are interested. they are in any book on trigonometry.

<u>Example</u>:



Show that

$\sin(30^\circ) = 1/2$

$\cos(30^\circ) = \sqrt{3}/2$

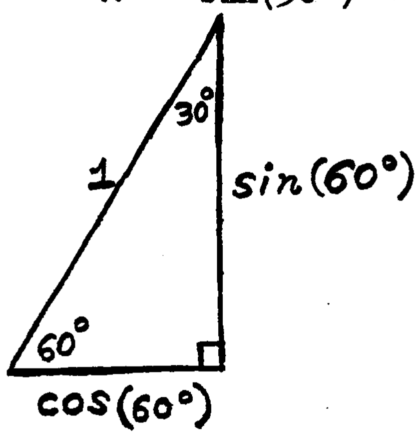
<u>Solution</u>: By flipping the triangle we see that

$$\sin(30^\circ) = \cos(60^\circ)$$

and

$$\begin{aligned}\cos(60^\circ) &= \cos(30^\circ+30^\circ)\\ &= \cos(30^\circ)\cos(30^\circ) - \sin(30^\circ)\sin(30^\circ)\\ &= \cos^2(30^\circ) - \sin^2(30^\circ)\\ &= (1 - \sin^2(30^\circ)) - \sin^2(30^\circ)\end{aligned}$$

$$\therefore \quad \sin(30^\circ) = 1 - 2\sin^2(30^\circ)\ .$$



Hence if $x = \sin(30^\circ) \implies$

$$x = 1 - 2x^2 \quad \underline{\text{or}}$$

$$2x^2 + x - 1 = 0$$

$$\implies x = \frac{-1 \pm \sqrt{1+8}}{2\cdot 2}$$

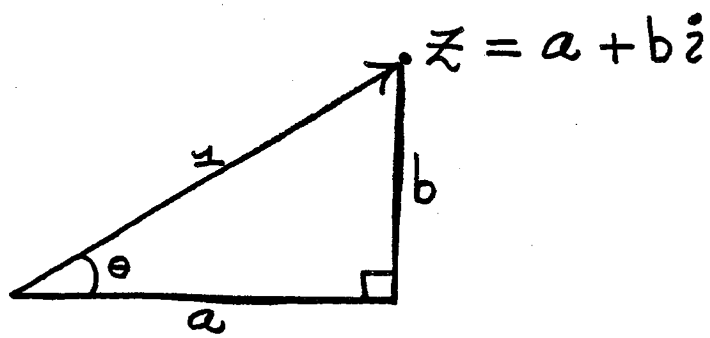
$$\implies x = \frac{-1 \pm 3}{4} = -1 \quad \text{or} \quad 1/2$$

$$\therefore \sin(30^\circ) > 0 \implies \boxed{\sin(30^\circ) = 1/2}$$

since $\sin^2(30^\circ) + \cos^2(30^\circ) = 1$

we get $\boxed{\cos(30^\circ) = \sqrt{3}/2}$.

Now let's consider complex numbers of <u>unit</u> <u>length</u>. These are of the form $z = a + bi$ with $a^2 + b^2 = 1$.



Thus we see that if θ is the angle that the vector of z makes with the x-axis, then

$$a = \cos(\theta)$$

$$b = \sin(\theta)$$

and $$\boxed{z = \cos(\theta) + i\sin(\theta)}.$$

With this help we can prove the following.

Theorem. The product of two complex numbers of unit length is a complex number of unit length. The angle of the product with the x-axis is the sum of the angles of the factors.

In other words,

$$[\cos(\theta) + i\sin(\theta)][\cos(\phi) + i\sin(\phi)]$$

$$\|$$

$$[\cos(\theta+\phi) + i\sin(\theta+\phi)] \quad .$$

The same result is true about the angles for the product of any two complex numbers. In general, the length of the product is the product of the lengths.

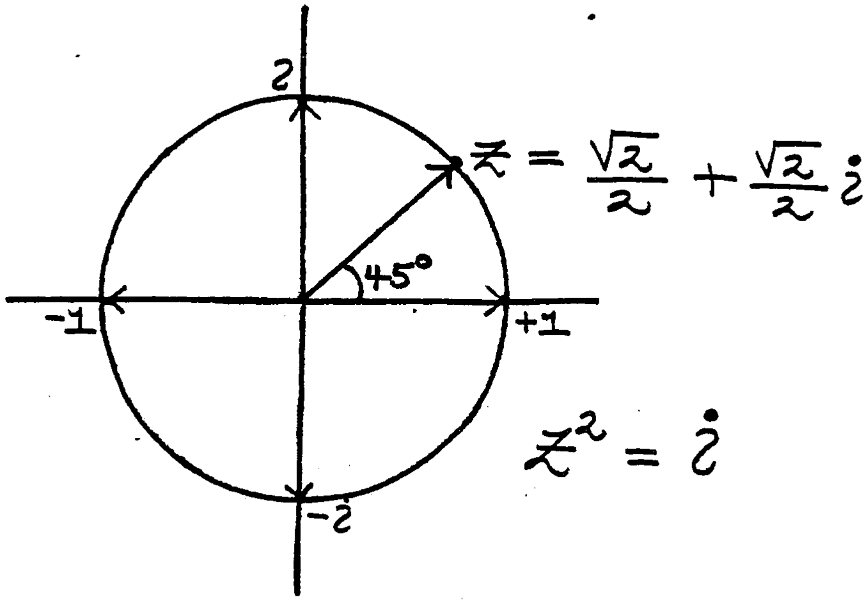
Example: $z = \cos(45^\circ) + i\sin(45^\circ) = \frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}$. Therefore we expect that

$$\begin{aligned} z^2 = zz &= \cos(45^\circ+45^\circ) + i\sin(45^\circ+45^\circ) \\ &= \cos(90^\circ) + i\sin(90^\circ) \\ &= 0 + i\cdot 1 \end{aligned}$$

$$\underline{z^2 = i}$$

Indeed: $\left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right)$

$$= \frac{\sqrt{2}^2}{4} + 2i\,\frac{\sqrt{2}^2}{4} - \frac{\sqrt{2}^2}{4} = 2i\left(\frac{2}{4}\right) = i$$



<u>Here is the proof of the theorem</u>:

$[\cos(\theta) + i \sin(\theta)][\cos(\phi) + i \sin(\phi)]$

$= [\cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)] + i[\sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)]$

(by multiplying these complex numbers)

$= [\cos(\theta+\phi) + i \sin(\theta+\phi)]$

(by using the trigonometry formulas for the addition of angles).

Thus the result is just an expression of the angle addition formulas, using complex numbers.

We leave it to you to see that the length of the product is the product of the lengths. (<u>HINT</u>: You can write any $z \neq 0$ in the form $z = RW$ where $R > 0$ and W has unit length.) (Or see Exercise 11.)

For example $z = 3 + 4i = \sqrt{3^2+4^2}\left(\frac{3}{\sqrt{3^2+4^2}} + \frac{4}{\sqrt{3^2+4^2}}\,i\right)$

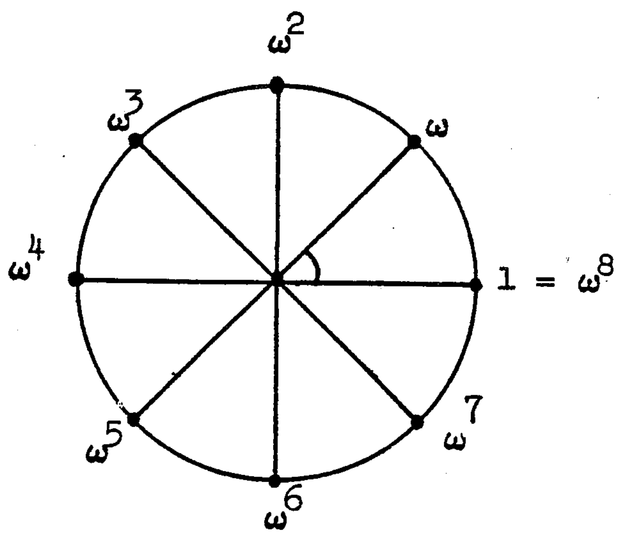
$$= 5\left(\tfrac{3}{5} + \tfrac{4}{5}\,i\right)$$

and length of $\left(\frac{3}{5} + \frac{4}{5}\,i\right)$:

$$\sqrt{\left(\tfrac{3}{5}\right)^2 + \left(\tfrac{4}{5}\right)^2} = \sqrt{\frac{3^2+4^2}{25}} = \sqrt{1} = 1.$$

One of the consequences of this result on the geometry of complex multiplication is that we can solve the equation $z^n = 1$ for <u>n different roots</u>. <u>This is done as follows</u>:

Let $\omega = \cos\left(\frac{360^\circ}{n}\right) + i\sin\left(\frac{360^\circ}{n}\right)$.



Here $\omega = \cos\left(\frac{360^\circ}{8}\right) + i\sin\left(\frac{360^\circ}{8}\right)$

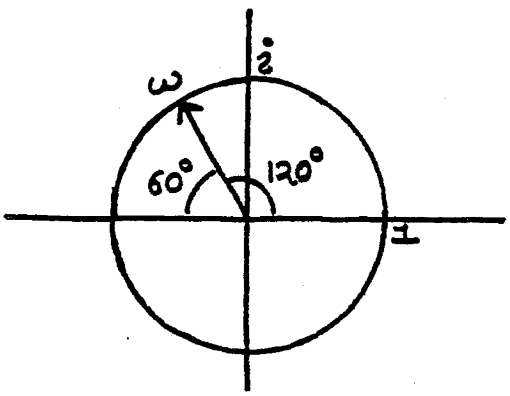
$$= \cos(45^\circ) + i\sin(45^\circ)$$

$$\omega = \frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}$$

Then $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are the n-distinct solutions to $z^n = 1$. Our geometry of multiplication tells us that $1, \omega, \omega^2, \ldots$ are equally spaced around the circle by increments of the angle $360^\circ/n$.

<u>Example</u>: Find all solutions to the equation $z^3 = 1$.

<u>Solution</u>: Let



$$\omega = \cos\left(\frac{360^\circ}{3}\right) + i\sin\left(\frac{360^\circ}{3}\right)$$

$$= \cos(120^\circ) + i\sin(120^\circ)$$

$$= -\cos(60^\circ) + i\sin(60^\circ)$$

$$\boxed{\omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2}}$$

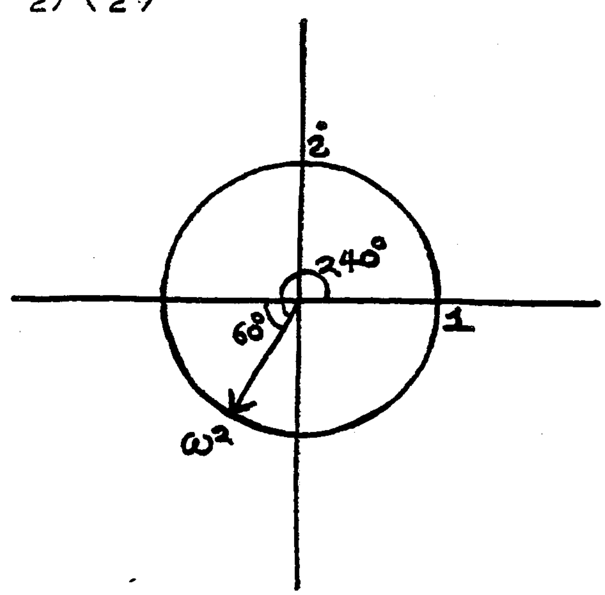
<u>Let's determine ω^2 in two ways</u>:

a) $$\omega^2 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

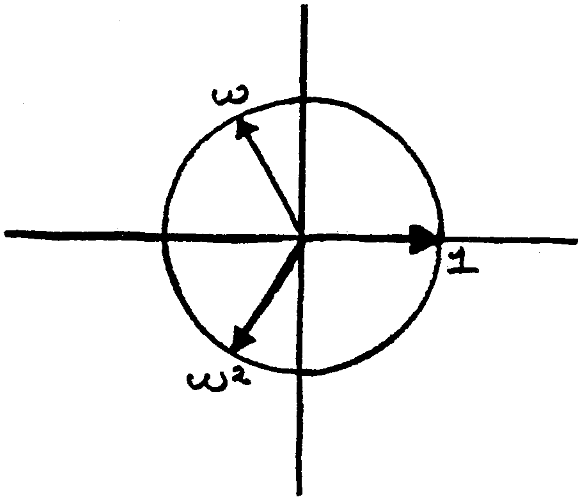
$$= \left(-\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2 + 2\left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)i$$

$$= \frac{1}{4} - \frac{3}{4} - \frac{\sqrt{3}}{2}i$$

$$\boxed{\omega^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i}\ .$$



b) By the theorem



$$\omega^2 = \omega\omega = \cos\left(\frac{360^\circ}{3} + \frac{360^\circ}{3}\right) + i\sin\left(\frac{360^\circ}{3} + \frac{360^\circ}{3}\right)$$

$$= \cos(240^\circ) + i\sin(240^\circ)$$

$$= -\cos(60^\circ) - i\sin(60^\circ)$$

$$\therefore \quad \omega^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

(Note that we got the same result!)

Thus the solutions to the equation $z^3 = 1$ are

$$1, \underbrace{\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)}_{\omega}, \underbrace{\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)}_{\omega^2}$$

<u>Hence</u>

$$\underline{z^3 - 1 = (z-1)(z-\omega)(z-\omega^2)}.$$

<u>Exercise</u> <u>12</u>. Find all solutions to the equation $z^6 = 1$.

Gauss proved the <u>Fundamental</u> <u>Theorem</u> <u>of</u> <u>Algebra</u> which says that any polynomial equation $a_nz^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0 = 0$

has solutions in the complex numbers. If $a_n \neq 0$ there will be n solutions (sometimes counted with multiplicity as in $(x-3)^2 = 0$).

Our example of $z^n - 1 = 0$ is a special case of Gauss' Theorem.

Exercise 13. a) Find all solutions to $z^3 = -1$. (HINT: if $x^3 = 1$ then $(-x)^3 = -1$.)

b) Solve $z^3 = 2$. (HINT: If $x^3 = 1$, then $(\sqrt[3]{2}\,x)^3 = 2$.)

Another special case of Gauss' Theorem is the quadratic formula. which exhibits the roots of a quadratic:

$$ax^2 + bx + c = 0$$

$$\Longrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

There are similar formulas for cubics and fourth power equations but not for any higher degrees. This negative result (and many positive consequences) was proved by Evariste Galois (in the 1700's). Galois died in a duel at the age of 21, but in his short life, left a legact of mathematics that continues to bear fruit even to the present day.

Exercise 14. (Solving the Cubic.)

a) Show that by setting $Y = X - a/3$. we may reduce the cubic equation

$$x^3 + ax^2 + bx + c = 0$$

to the form $Y^3 + PY + Q = 0$.

b) Reduce $X^3 + X^2 + X + 1 = 0$ to the form $Y^3 + PY + Q = 0$ via $Y = (X - 1/3)$ and determine P and Q.

c) Consider the cubic equation

$$X^3 + PX + Q = 0 \quad (P \text{ and } Q \text{ constants}).$$

Let $X = A + B$, substitute and show that we get

$$A^3 + B^3 + (3AB+P)(A+B) + Q = 0.$$

Thus we <u>would</u> get a solution if

$$A^3 + B^3 + Q = 0$$

<u>and</u>

$$3AB + P = 0.$$

But if $3AB + P = 0$, then $A^3B^3 = -P^3/27$. So we would have

$$\boxed{\begin{aligned} A^3 + B^3 &= -Q \\ A^3B^3 &= -P^3/27 \end{aligned}}\,.$$

Knowing P and Q we can solve for A^3 and B^3 since they are the roots of the quadratic $x^2 + Qx - P^3/27 = 0$. Hence we can take cube roots of the solutions to this quadratic, add them up and get solutions to the cubic. <u>This is an outline of a general method for solving the cubic</u>.

<u>Example</u>: $X^3 - 6X - 6 = 0.$

$P = -6,\ Q = -6$

$\therefore -Q = 6$

$$-P^3/27 = -\frac{(-2.3)^3}{3^3} = 8.$$

So we have $A^3 + B^3 = -Q = 6$

$$A^3B^3 = -P^3/27 = 8.$$

Hence A^3, B^3 are the solutions to

$$x^2 - 6x + 8 = 0$$

$$(x-2)(x-4) = 0.$$

So we take $A^3 = 2,\ B^3 = 4.$

Letting $\omega = \cos(120^\circ) + i\sin(120^\circ) = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$ we have

$$A = 2^{\frac{1}{3}},\ 2^{\frac{1}{3}}\omega,\ 2^{\frac{1}{3}}\omega^2$$

and

$$B = 4^{\frac{1}{3}},\ 4^{\frac{1}{3}}\omega,\ 4^{\frac{1}{3}}\omega^2.$$

Hence the solutions $x = A + B$ of $x^3 - 6x - 6 = 0$ are

$x = 2^{\frac{1}{3}} + 4^{\frac{1}{3}}$ or $2^{\frac{1}{3}}\omega + 4^{\frac{1}{3}}\omega^2$ or $2^{\frac{1}{3}}\omega^2 + 4^{\frac{1}{3}}\omega$. (Note that we need $AB = -\frac{P}{3}$ <u>real</u> Since the original conditions are $A^3 + B^3 = -Q = 6$ $3AB = -P = 6$.)

Problem: Check by direct substitution that $2^{\frac{1}{3}} + 4^{\frac{1}{3}}$ and $2^{\frac{1}{3}}\omega + 4^{\frac{1}{3}}\omega^2$ are solutions to $x^3 - 6x - 6 = 0$.

(HINT: You may use the fact that $\omega^3 = 1$ and that $1 + \omega + \omega^2 = 0$. Do not write out ω in its real and complex parts.)

Problem: Find the solutions to the cubic $x^3 - 6x - 9 = 0$. (HINT: Note that $x = 3$ is a solution. Hence you can check your answer by another route!)

Work this out using the method of solution that we have outlined.

Remark: Any cubic equation has real roots, but the problem of solving the cubic gave great impetus to the development of complex numbers! There are many cases where the only way to get a formula for the real root of a cubic involves using imaginary values. This will happen when the associated quadratic equation has imaginary roots. (Try solving $x^3 - 3x - 1 = 0$.)

This ends our journey into some of the intricacies of polynomial equations and complex numbers. For a return to the ideas in part I you might enjoy the following iterative scheme for solving a cubic (for a real root):

$$\boxed{x^3 = Rx + S} \qquad (R = -P,\ S = -Q \text{ above})$$

$$\Longrightarrow \quad 2x^3 = x^3 + Rx + S$$

$$\frac{2x^3}{x^2} = \frac{x^3}{x^2} + \frac{Rx}{x^2} + \frac{S}{x^2}$$

$$2x = x + R/x + S/x^2$$

$$x = \tfrac{1}{2}(x + R/x + S/x^2) = f(x)$$

To get a sequence of numbers converging on a root of $x^3 = Rx + S$ take a constant $\ast$ and compute

$$f(\ast),\ ff(\ast),\ fff(\ast), \ldots$$

For example, if you try this for

$$f(x) = \tfrac{1}{2}(x + 6/x + 6/x^2)$$

you get

$$\begin{aligned}
f(1) &= \tfrac{1}{2}(1 + 6 + 6) = \tfrac{13}{2} = 6.5 \\
f^2(1) &= 3.7825 \\
f^3(1) &= 2.8940 \\
f^4(1) &= 2.8418 \\
f^5(1) &= 2.8480 \\
f^6(1) &= 2.8472 \\
f^7(1) &= 2.8473 \\
f^8(1) &= 2.8473
\end{aligned}$$

So 2.8473 should be an approximate real root of $x^3 - 6x - 6 = 0$. We know that $2^{1/3} + 4^{1/3}$ <u>is</u> a real root and in fact, you can check that this is a good approximation to $2^{1/3} + 4^{1/3}$.

$F(x) = \frac{1}{2}(x + A/x^2)$ will iterate to produce $A^{1/3}$.

$$2^{\frac{1}{3}} = 1.25992105\ldots$$

$$\underline{4^{\frac{1}{3}} = 1.587401052\ldots}$$

$$2^{\frac{1}{3}} + 4^{\frac{1}{3}} = 2.847322102\ldots$$

FINIS

SUPPLEMENTARY PROBLEMS

The following paragraphs are each a cross between a problem and some new information. In each case you can have a good time thinking through the material and trying out variations on these themes.

1. $\boxed{e^{i\pi} + 1 = 0}$

This is one of the great mysterious equations of mathematics. You know all of its symbols, $i = \sqrt{-1}$, $\pi = 3.14159\ldots$, except perhaps for e. e is also a real number. Its value is approximately $e = 2.7182818\ldots$ More precisely, $e = \lim\limits_{n\to\infty} (1+\frac{1}{n})^n$. For example, you can check on a calculator that $(1.01)^{100} \approx 2.7$, a good first approximation. This number e is very special; it comes up in studying compound interest, logarithms <u>and</u> it is fundamental to understanding complex numbers.

One can see that

$$e^x = \lim_{n\to\infty}(1 + x/n)^n.$$

Do you believe this? Why?

Because there are good ways to evaluate this limit, e is used as the base of the natural logarithms. Another, and computationally useful, version of e^x is given by the formula

$$\boxed{e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots}$$

Here

$$1! = 1$$

$$2! = 1\cdot 2 = 2$$

$$3! = 1\cdot 2\cdot 3 = 6$$

$$4! = 1\cdot 2\cdot 3\cdot 4 = 24$$

$$5! = 1\cdot 2\cdot 3\cdot 4\cdot 5 = 120$$

$$6! = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6 = 720$$

etc...

<u>Exercise</u>: Use the above formula to get an approximation for e by computing

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} \quad .$$

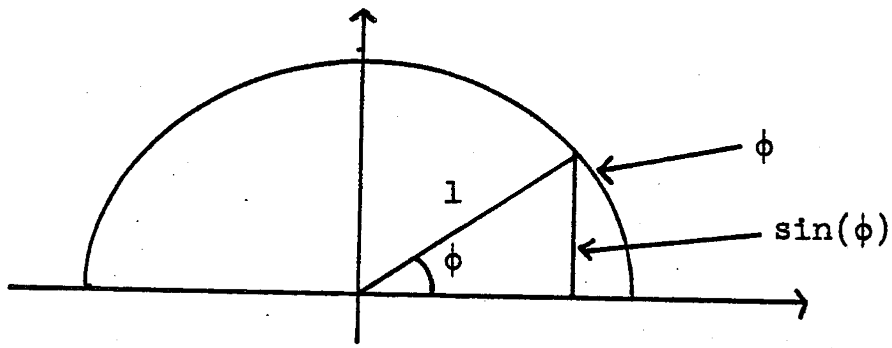
If we allow e to have an <u>imaginary exponent</u>, then life gets even more interesting:

<u>Theorem</u>: (<u>Euler's Formula</u>)

$$\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta)}$$

We won't prove this formula, but we can easily show you how and why it works: Consider $1 + (\theta/n)i$ for fixed θ and <u>large n</u>. For n very large, θ/n is very small and hence

$$\cos(\theta/n) \approx 1$$
$$\sin(\theta/n) \approx \theta/n.$$



In radian measure $\sin(\phi) \approx \phi$ for small ϕ. In radian measure the angle is the length of the arc on a circle of radius 1. Thus $360^\circ = 2\pi$ radians.

Hence $\left(1 + \frac{i\theta}{n}\right) \approx \cos\left(\frac{\theta}{n}\right) + i \sin\left(\frac{\theta}{n}\right)$.

<u>Therefore</u>

$$e^{i\theta} \approx \left(1 + \frac{i\theta}{n}\right)^n \quad \left[\text{since } e^x \approx \left(1 + \frac{x}{n}\right)^n \text{ large } n\right]$$

$$\approx \left(\cos\left(\frac{\theta}{n}\right) + i \sin\left(\frac{\theta}{n}\right)\right)^n$$

$$\|$$

$$\cos(\theta) + i \sin(\theta) \qquad \text{[Why?]}$$

$$\therefore \quad e^{i\theta} \approx \cos(\theta) + i \sin(\theta).$$

This approximation gets better and better for large n and allows one to conclude the Theorem.

<u>Exercise</u>: Explain how the formula

$$e^{i\pi} + 1 = 0$$

can be deduced from the Euler formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. (Let $\theta = \pi$.)

<u>Exercise</u>: Show

$$\sqrt{-1}^{\sqrt{-1}} = e^{-\pi/2}.$$

---

2. Work out a <u>formula</u> for solving the cubic $x^3 + Px + Q = 0$.

---

3. If $x^2 = ax + b$, then iteration suggests that

$$x = a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \cdots}}}$$

In the case of real roots, which root does this represent? (Do some calculations). Note that here we are iterating

$$f(x) = a + \frac{b}{x}.$$

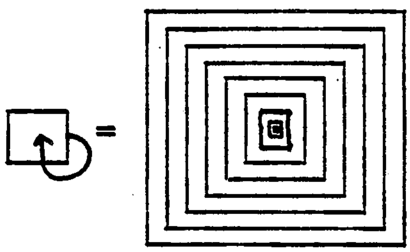
---

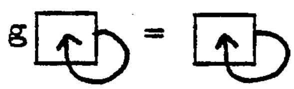
4. If $g(x) = \boxed{x}$

then $g^2(x) = \boxed{\boxed{x}}$

$g^3(x) = \boxed{\boxed{\boxed{x}}}$

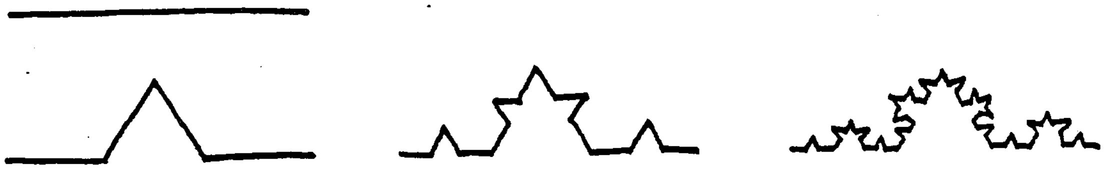
$g^4(x) = \boxed{\boxed{\boxed{\boxed{x}}}}$

and if  (infinitely many boxes) (inside)

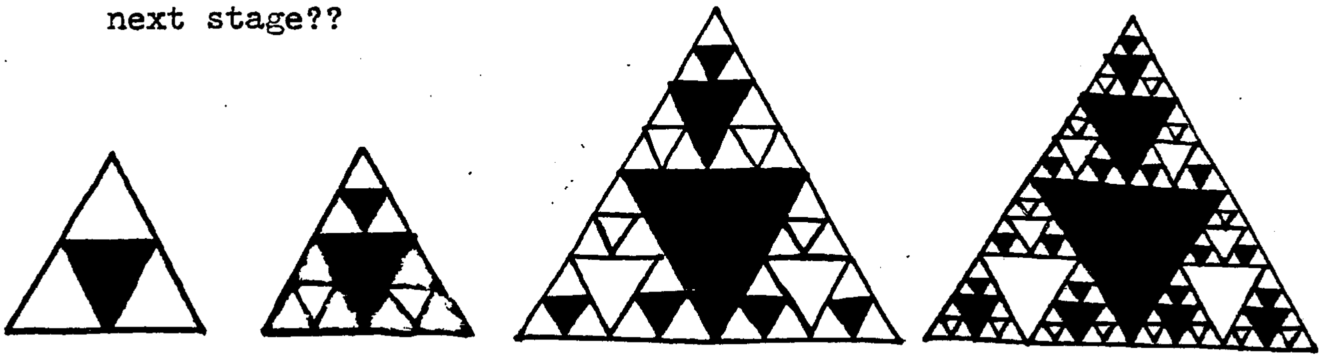
then 

Try iterating $F(x) = \boxed{x \; \boxed{x}}$.

What does an "infinity box" solution to $F(x) = x$ look like? Try drawing a picture of it.

5. 

By what rule are we generating these pictures? What is the next stage??



6. If $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, then

$$\cos(\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

$$\sin(\theta) = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

What happens to these formulas if you write

$$e^{i\theta} = 1 + \frac{(i\theta)}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \dots ?$$

Note that since $i = i,\ i^2 = -1,\ i^3 = -i,\ i^4 = 1$

$\Longrightarrow i^5 = i$, etc...

Thus
$$\begin{aligned} e^{i\theta} = 1 &- \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} \pm \dots \\ &+ \frac{i\theta}{1!} - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} \pm \dots \end{aligned}$$

This suggests that

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} \pm \cdots\right)$$

$$+ i\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} \pm \cdots\right)$$

and hence we are led to guess that

$$\boxed{\begin{aligned} \cos(\theta) &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} \pm \cdots \\ \sin(\theta) &= \frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} \pm \cdots \end{aligned}}$$

These series formulas for sin and cos are correct! You can check them by taking the first few terms from each series and comparing the result with what you know or with the trig functions on your calculator.

<u>EXAMPLE</u>: We know $\sin(45^\circ) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = .707\ldots$ Now

$$\frac{(\pi/4)}{1!} - \frac{(\pi/4)^3}{3!} + \frac{(\pi/4)^5}{5!}$$

$$\|$$

$$\frac{.78539}{1} - \frac{.48447}{6} + \frac{.29884}{120}$$

$$\|$$

$$.78536 \qquad .08074 + .00249$$

$$\|$$

$$.70711$$