Discrete Calculus

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1 Introduction

How many times have you wanted to know a good reason that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. Sure, it's true by induction, but how in the world did we get this formula? Or $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$? Well, there are several ways to arrive at these conclusions, but Discrete Calculus is one of the most beautiful. Recall (or just nod along) that in normal calculus, we have the derivative and the integral, which satisfy some important properties, such as the fundamental theorem of calculus. Here, we create a similar system for discrete functions.

2 The Discrete Derivative

We define the discrete derivative of a function $f(n)$, denoted $\Delta_n f(n)$, to be $f(n+1) - f(n)$. This operator has some interesting properties.

2.1 Properties

- Linear: $\Delta_n(f+g)(n) = \Delta_n f(n) + \Delta_n g(n)$, $\Delta_n(cf(n)) = c\Delta_n f(n)$
- Product rule: $\Delta_n(f \cdot g)(n) = f(n+1)\Delta_n g(n) + \Delta_n f(n) g(n)$
- Quotient rule: $\Delta_n(f/g)(n) = \frac{\Delta_n f(n) g(n) - f(n)\Delta_n g(n)}{g(n)g(n+1)}$

3 Basic Differentiation

Remember that in standard calculus, $\frac{d}{dx}(x^n) = nx^{n-1}$. We want something similar in discrete calculus. Consider the "fallling power" $n^{\underline{k}} = n(n-1)(n-2)(n-3)(\cdots)(n-k+1)$. $\Delta_n(n^{\underline{k}}) = (n+1)(n)(n-1)(n-2)(\cdots)(n-k+2) - n(n-1)(n-2)(\cdots)(n-k+2)(n-k+1) = n(n-1)(n-2)(\cdots)(n-k+2)(n+1-(n-k+1)) = kn^{\underline{k-1}}$. This acts just like our normal power for calculus purposes!

So, we can differentiate stuff such as $n^2 = (n^2 - n) + n = n^{\underline{2}} + n^{\underline{1}}$, so $\Delta_n(n^2) = 2n^{\underline{1}} + n^{\underline{0}} = 2n + 1$. Notice that this is exactly what we get if we just do $\Delta_n(n^2) = (n+1)^2 - n^2 = 2n + 1$.

3.1 The Discrete *e*

e is the number such that $\frac{d}{dx}(e^x) = e^x$. It follows, that the discrete e should follow that $\Delta_n(e^n) = e^n$. It turns out that this e is 2, as $2^{n+1} - 2^n = 2^n$, which is exactly what we wanted. We also have that $\Delta_n(a^n) = a^{n+1} - a^n = (a-1)a^n$.

4 The Discrete Integral

We now want something similar to the integral, which is simply summation. $\sum\limits_{n:a\to b} f(n) = f(a) + f(a+1) + \cdots + f(b-2) + f(b-1)$. Notice that it does not include $f(b)$, which can be confusing as the normal summation, $\sum\limits_{n=a}^{b} f(n)$ *does* include $f(b)$. This integral is also linear, and it follows the fundamental theorem: $\sum\limits_{n:a\to b} \Delta_n f(n) = f(b) - f(a)$. This can be easily seen by expanding the sum.

5 Basic Integration

Since we have the fundamental theorem, it's easy to determine that $\sum\limits_{n:a\to b} n^{\underline{k}} = \frac{b^{\underline{k+1}} - a^{\underline{k+1}}}{k+1}$. This immediately gives us things such as $\sum\limits_{n:1\to a} n^2 = \sum\limits_{n:1\to a} n^{\underline{2}} + n^{\underline{1}} = \frac{a^{\underline{3}} - 1^{\underline{3}}}{3} + \frac{a^{\underline{2}} - 1^{\underline{2}}}{2} = \frac{a(a-1)(a-2)}{3} + \frac{a(a-1)}{2} = \frac{a(a-1)(2a-4) + 3a(a-1)}{6} = \frac{a(a-1)(2a-1)}{6}$. But remember that this is actually the sum up to $(a-1)^2$. It then follows that $\sum\limits_{i=1}^{a} i^2 = \frac{a(a+1)(2a+1)}{6}$.

6 Summation by Parts

We now want to find something similar to integration by parts, so we play with the product rule a bit. $\Delta_n(f\cdot g)(n) - f(n+1)\Delta_n g(n) = g(n)\Delta_n f(n)$. Summing both sies, we obtain $\sum\limits_{n:a\to b} g(n)\Delta_n f(n) = f(b)g(b) - f(a)g(a) - \sum\limits_{n:a\to b} f(n+1)\Delta_n g(n)$

This allows us to do more advanced summations such as $\sum_{n=0}^{k} n2^n$. We simply apply parts to get $\sum_{n:0\to k} n2^n = k2^k - 0\cdot 2^0 - \sum_{n:0\to k} 2^{n+1}\cdot 1 = k2^k - (2^{k+1}-2) = (k-2)2^k + 2$, or $\sum_{n=0}^{k} n2^n = (k-1)2^{k+1}+2$. It'd be very difficult to do such things without motivation like this.

7 Finite Differences

Problem: Find an 4-degree polynomial $p(n)$ such that $p(0) = 1$, $p(1) = 4$, $p(2) = 57$, $p(3) = 232$, and $p(4) = 625$.

How would you tackle such a problem? You could plug these values into something like Lagrange Interpolation, but who memorizes that? (The answer is lots of people, but that's beside the point) Discrete Calculus gives us a very nice way to do such a thing. This is called finite differences. Make a table with the values of $\Delta_n^i p(n)$ for $i = 0, 1, 2, 3, 4$, like this:

1	4	57	232	625
3	53	175	393	
50	122	218		
72	96			
24				

Aha! I know what the last row is! $\Delta_n^4 p(n) = 24$. Now we simply integrate with the appropriate constant to get the remaining ones. The appropriate constant happens to be the first number in each row starting from the bottom.

$$\begin{aligned}
\Delta_n^4 p(n) &= 24 \\
\Delta_n^3 p(n) &= 24n^{\underline{1}} + 72 \\
\Delta_n^2 p(n) &= 12n^{\underline{2}} + 72n^{\underline{1}} + 50 \\
\Delta_n p(n) &= 4n^{\underline{3}} + 36n^{\underline{2}} + 50n^{\underline{1}} + 3 \\
p(n) &= n^{\underline{4}} + 12n^{\underline{3}} + 25n^{\underline{2}} + 3n^{\underline{1}} + 1
\end{aligned}$$

So, $p(n) = n^{\underline{4}} + 12n^{\underline{3}} + 25n^{\underline{2}} + 3n^{\underline{1}} + 1 = n(n-1)(n-2)(n-3) + 8n(n-1)(n-2) + 25n(n-1) + 3n + 1 =$

$(n^4-6n^3+11n^2-6n)+12(n^3-3n^2+2n)+25(n^2-n)+3n+1 = n^4+6n^3-4n+1$, which indeed fits our data points.