0. Notational Warning. In these notes $x^k$ equals the k-th power of $x$, but $x^{(k)} = x(x-1)(x-2)...(x-k+1)$. Thus

$x(0) = 1$
$x(1) = x$
$x(2) = x(x-1)$
$x(3) = x(x-1)(x-2)$
and so on.

1. We are given a function $f(n)$, defined on the natural numbers, possibly including 0 in its domain.

2. Define the discrete derivative (difference operator) by the formula

$$\Delta f(n) = f(n+1) - f(n).$$

For example, if $f(n) = n^2$ then $\Delta f(n) = (n+1)^2 - n^2 = 2n + 1$.

3. Note that if $n^{(k)} = n(n-1)(n-2)...(n-k+1)$, then

$$\Delta n^{(k)} = k n^{(k-1)}.$$

For example,

$$\Delta n^{(3)} = \Delta [n(n-1)(n-2)]$$

$$= (n+1)(n)(n-1) - n(n-1)(n-2)$$

$$= n(n-1)((n+1)-(n-2))$$

$$= 3 n(n-1)$$

$$= 3 n^{(2)}.$$

4. Theorem. Suppose that $\Delta F(n) = \Delta G(n)$ for all $n = 0,1,2,...$, then $F(n) = G(n) + k$ for a constant $k$ that is independent of $n$.

Proof. Let $k = F(0) - G(0)$. Then the theorem is true for $n = 0$. For $n=1$ we have $F(1) - F(0) = G(1) - G(0)$ since $\Delta F(0) = \Delta G(0)$. 

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Hence \( F(1) - G(1) = F(0) - G(0) = k \). This proves the Theorem for \( n = 1 \). Now suppose we have proved the Theorem for \( n \leq N \). That is, we assume that for \( n \leq N \), \( F(n) - G(n) = k \).

Then, since \( \Delta F(N) = \Delta G(N) \), we have \( F(N+1) - F(N) = G(N+1) - G(N) \).

Hence \( F(N+1) - G(N+1) = F(N) - G(N) = k \). This completes the proof of the Theorem by induction. //

5. **Problem:** Suppose \( \Delta F(n) = n^2 \). Find all such \( F(n) \).

**Solution.** We know \( \Delta n(3) = 3n(n-1) = 3n^2 - 3n \).

And we know that \( \Delta n(2) = 2n(1) = 2n \). Thus

\[ \Delta [ \frac{1}{3}n(3) + \frac{1}{2}n(2)] = n^2. \]

Therefore \( F(n) = \frac{1}{3}n(3) + \frac{1}{2}n(2) + k \) for any constant \( k \). //

6. **Problem.** Find a formula for \( G(n) = 1^2 + 2^2 + 3^2 + \ldots + n^2 \).

**Solution.** Clearly, \( \Delta G(n) = (n+1)^2 \). Therefore \( G(n) = F(n+1) \) in the previous problem. Thus

\[ G(n) = \frac{1}{3}(n+1)(3) + \frac{1}{2}(n+1)(2) + k \]

and \( G(1) = 1 \). But

RHS at \( k=1 \)

\[ = \frac{1}{3}(2)(3) + \frac{1}{2}(2)(2) + k \]

\[ = \frac{1}{3}(2(1)(0) + (1/2)(2(1)) + k \]

\[ = 1 + k. \]

Therefore \( k = 0 \), and we conclude that

\[ 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{3}(n+1)(3) + \frac{1}{2}(n+1)(2) \]

This completes the solution. //

7. **Exercise.** Use the formula at the end of 6. to show that

\[ 1^2 + 2^2 + 3^2 + \ldots + n^2 = (1/6)n(n+1)(2n+1) \] for all \( n= 1,2,3,\ldots \)

8. **Exercise.** Find a formula for

\[ H(n) = 1^4 + 2^4 + 3^4 + \ldots + n^4. \]
Here are the background calculations that will let you solve Exercise 8 and other problems of this type. What we are going to do is like making a table of integrals. We cannot immediately see the answer to the discrete integral of $n^k$, but we do know that the discrete integral of $n(k)$ is $n(k+1)/(k+1)$. So the strategy is to write $n^k$ in terms of terms of the form $n(r)$.

A. $n = n^1 = n(1)$

B. $n^2 = n(n-1) + n = n(2) + n(1)$

C. $n^3 = n(3) + 3n(2) + n(1)$

D. $n^4 = n(4) + 6n(3) + 7n(2) + n(1)$

These formulas are obtained by writing out $n(r)$ for $r=1,2,3,4$. For example, in B. we write $n(2) = n(n-1) = n^2 - n$, so $n^2 = n(2) + n = n(2) + n(1)$.

Then

$n(3) = n(n-1)(n-2) = n^3 - 3n^2 + 2n$.

The formula in C follows by rewriting this as a formula for $n^3$ and substituting the results in B and in A. Similarly for part D.