Discrete Calculus

by Louis H. Kauffman

0. **Notational Warning.** In these notes x^k equals the k-th power of x, but $x^{(k)} = x(x-1)(x-2)...(x-k+1)$. Thus

$x^{(0)} = 1$

$x^{(1)} = x$

$x^{(2)} = x(x-1)$

$x^{(3)} = x(x-1)(x-2)$

and so on.

1. We are given a function f(n), defined on the natural numbers, possibly including 0 in its domain.

2. Define the *discrete derivative* (difference operator) by the formula

$$\Delta f(n) = f(n+1) - f(n).$$

For example, if $f(n) = n^2$ then $\Delta f(n) = (n+1)^2 - n^2 = 2n + 1$.

3. Note that if $n^{(k)} = n(n-1)(n-2)...(n-k+1)$, then

$$\Delta n^{(k)} = k\, n^{(k-1)}.$$

For example,

$$\Delta n^{(3)} = \Delta[\, n(n-1)(n-2)\,]$$

$$= (n+1)(n)(n-1) - n(n-1)(n-2)$$

$$= n(n-1)[(n+1) -(n-2)]$$

$$=3\, n(n-1)$$

$$= 3\, n^{(2)}.$$

4. **Theorem.** Suppose that $\Delta F(n) = \Delta G(n)$ for all n = 0,1,2,..., then $F(n) = G(n) + k$ for a constant k that is independent of n.

Proof. Let $k = F(0) - G(0)$. Then the theorem is true for n =0.
For n=1 we have $F(1) - F(0) = G(1) - G(0)$ since $\Delta F(0) = \Delta G(0)$.

Hence $F(1) - G(1) = F(0) - G(0) = k$. This proves the Theorem for $n = 1$. Now suppose we have proved the Theorem for $n <= N$. That is, we assume that for $n <= N$, $F(n) - G(n) = k$.
Then, since $\Delta F(N) = \Delta G(N)$, we have $F(N+1) - F(N) = G(N+1) - G(N)$.
Hence $F(N+1) - G(N+1) = F(N) - G(N) = k$. This completes the proof of the Theorem by induction. //

5. **Problem:** Suppose $\Delta F(n) = n^2$. Find all such $F(n)$.

Solution. We know $\Delta n^{(3)} = 3\, n(n-1) = 3n^2 - 3n$.
And we know that $\Delta n^{(2)} = 2\, n^{(1)} = 2n$. Thus

$$\Delta[\ (1/3)n^{(3)} + (1/2)n^{(2)}\ ] = n^2.$$

Therefore $F(n) = (1/3)n^{(3)} + (1/2)n^{(2)} + k$ for any constant k. //

6. **Problem.** Find a formula for $G(n) = 1^2 + 2^2 + 3^2 + ... + n^2$.

Solution. Clearly, $\Delta G(n) = (n+1)^2$. Therefore $G(n) = F(n+1)$ in the previous problem. Thus

$$G(n) = (1/3)(n+1)^{(3)} + (1/2)(n+1)^{(2)} + k$$

and $G(1) = 1$. But
RHS at k=1
$= (1/3)(2)^{(3)} + (1/2)(2)^{(2)} + k$
$= (1/3)(2(1)(0)) + (1/2)(2(1)) + k$
$= 1 + k$.

Therefore $k = 0$, and we conclude that

$$1^2 + 2^2 + 3^2 + ... + n^2 = (1/3)(n+1)^{(3)} + (1/2)(n+1)^{(2)}$$

This completes the solution.//

7. **Exercise.** Use the formula at the end of 6. to show that

$$1^2 + 2^2 + 3^2 + ... + n^2 = (1/6)n(n+1)(2n+1) \text{ for all } n= 1,2,3,....$$

8. **Exercise.** Find a formula for

$$H(n) = 1^4 + 2^4 + 3^4 + ... + n^4.$$

9. **Remarks.**

Here are the background calculations that will let you solve Exercise 8 and other problems of this type. What we are going to do is like making a table of integrals. We cannot immediately see the answer to the discrete integral of n^k, but we do know that the discrete integral of $n^{(k)}$ is $n^{(k+1)}/(k+1)$. So the strategy is to write n^k in terms of terms of the form $n^{(r)}$.

A. $n = n^1 = n^{(1)}$

B. $n^2 = n(n-1) + n = n^{(2)} + n^{(1)}$

C. $n^3 = n^{(3)} + 3n^{(2)} + n^{(1)}$

D. $n^4 = n^{(4)} + 6n^{(3)} + 7n^{(2)} + n^{(1)}$

These formulas are obtained by writing out $n^{(r)}$ for r=1,2,3,4.
For example, in B. we write $n^{(2)} = n(n-1) = n^2 - n$, so
$n^2 = n^{(2)} + n = n^{(2)} + n^{(1)}$.
Then
$n^{(3)} = n(n-1)(n-2) = n^3 - 3n^2 + 2n$.
The formula in C follows by rewriting this as a formula for n^3 and substituting the results in B and in A.
Similarly for part D.