Wang Algebra and Matroids

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***Abstract*—Wang algebra is defined by three rules: i) $xy = yx$; ii) $x + x = 0$; and iii) $xx = 0$. K. T. Wang showed that these rules give a shortcut method for finding the joint resistance (or driving point resistance) of an electrical network. However, there are electrical systems more general than the Kirchhoff network. For these systems regular matroids replace networks. It is shown in this paper that Wang algebra is an excellent tool to develop properties of networks. Moreover the Wang shortcut method can still be used to find the joint resistance of an electrical network.**

I. Network Discriminants

KIRCHHOFF gave himself the problem of determining the joint resistance of an electrical network. He found a formula for the joint resistance as a ratio of two determinants having a very special form. These determinants are homogeneous multilinear polynomial functions of the branch resistance. The polynomials have the remarkable property in that the coefficients of all of the terms have the value $+1$. Such determinants are termed *unimodular discriminants*.

Later Maxwell, in his well-known treatise, gave a different analysis of the network problem. His treatment is based on branch conductances rather than branch resistances. This led to the joint resistance being represented as a ratio of unimodular discriminants in the branch conductances. This was pointed out in a footnote by J. J. Thompson.

In 1934, K. T. Wang found simple algebraic rules which directly determine the unimodular discriminants [10]. Thus it is unnecessary to go through the tedious process of formulating Kirchhoff's equations. Wang's method is a rapid method of determining the joint resistance as a function of the parameters, especially for hand calculation. (Optimal numerical procedures, when the parameters are fixed rather than variable, are an entirely different problem.)

There are electrical systems more general than the classical Kirchhoff network. The question then arises as to whether or not the joint resistance is a ratio of unimodular discriminants. The answer is yes, if the Kirchhoff graph is replaced by a generalization termed a *regular matroid*. The branch resistors are assigned to quasi-circuits of the matroid. Then applying Kirchhoff's voltage drop law to the quasi-circuits leads to a system of equations which determine the current in the system. Moreover, the Wang rules can be used to evaluate the joint resistance function.

The main goal of this paper is to give a simple definition and analysis of matroids. This proves possible by using the Wang algebra as a tool.

Before proceeding to the matroid generalization we show in Section II how to apply Wang algebra to compute network functions for the classical Kirchhoff network. Wang's original method used the mesh formulation of Kirchhoff's laws. In Section II we apply the Wang algebra to the nodal formulation of Kirchhoff's laws. As a corollary it is shown how to evaluate any symmetric determinant by Wang's three rules: i) $xy = yx$; ii) $x + x = 0$; and iii) $xx = 0$.

The Wang algebra is considered abstractly in Section III. It is found to be an ideal tool in analyzing linear dependence and independence in a vector space U with mod 2 scalars. Since U is a vector space over the two element field (sometimes denoted $\mathbb{Z}_2$ or $GF(2)$), the only scalars we are allowed to multiply by vectors are the scalars 0 and 1. The subspaces of U are shown to be characterized by Wang products of independent vectors.

The Wang product of vectors x and y is denoted as xy. We also introduce another product denoted as (x,y). These two products are termed *outer* and *inner*, respectively. The inner product (x,y) is the ordinary scalar product evaluated modulo 2. Thus (x,y) is either zero or one. This inner product is not positive definite but many of the usual properties of the ordinary scalar product continue to hold.

A *binary matroid* is defined as a pair (V,V') of a subspace V of U and subspace V' which is the orthogonal complement of V under the above defined inner product. Of course, since the inner product (x,y) is not positive definite $V \cap V'$ need not be equal to $\{0\}$. However many of the usual properties of the orthogonal complement remain true, for instance $\dim(U) = \dim(V) + \dim(V')$. We use the notation V' to distinguish this orthogonal complement from the standard orthogonal complement. The electric interpretation of V is the voltage space and the interpretation of V' is the current space. Thereby electrical duality is built into the theory from the beginning. The outer product of a set of basis vectors (it matters not which basis) of V is a network discriminant.

A tree of a matroid is defined to be a term in the outer (Wang) product of a set of basis vectors of V (it makes no difference which basis). A cotree is defined to be a term in the outer product of a basis of V'.

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Let $e_1, e_2, \cdots, e_m$ be the main basis vectors for the vector space U. In the case where the matroid is derived from a graph, with V being the cut space and V' being the mesh space, these basis vectors will correspond to the branches. The outer product for a set of n vectors may be regarded as a polynomial of degree n in the e_i's. The Wang trick is now to regard $e_1, \cdots, e_m$ as real variables. This polynomial is interpreted as the discriminant. The discriminant is of primary importance in studying matroids. For instance, we show that the discriminant is factorable into two polynomials of smaller degree if and only if the matroid is separable. This means that the matroid can be written as a direct sum of smaller matroids.

To obtain a real electrical correspondence it is necessary to assign a "direction" to the "edges" of the matroid. This means that the vectors of V now have some components with values other than 0 and 1. Thus the mod two vector space U is replaced by $\mathbb{R}^N$. When the Wang trick works for a subspace of $\mathbb{R}^N$, then the pair $(S, S^\perp)$ of S and its orthogonal complement then $(S, S^\perp)$ is termed a regular matroid; the orthogonal complement is taken relative to the standard inner product of $\mathbb{R}^N$. The theory of regular matroids is closely related to the theory of *totally unimodular matrices*. A regular matroid has a unimodular discriminant and it results that electrical systems based on regular matroids will be very similar to the classical networks based on graphs. The advantage of regular matroids over graphs is that regular matroids will without exception satisfy the principle of electrical duality. It is thought by some (see [11]) that the closer analysis of networks based on regular matroids may lead to insight to the solution of the resistive n-port problem.

II. Wang Algebra Applied to Graphs

An algebra with the property that

$$x + x = 0 \quad \text{and} \quad x \cdot x = 0 \tag{1}$$

for each element x of the algebra is termed a Wang algebra. This algebra gives an interesting method of determining the basic functions associated with an electrical network.

The application of the Wang algebra to networks can be illustrated by the problem of determining the joint resistance of the network shown in Fig. 1. The letters a, b, c, d, and k designate the five branches of the network. The numbers 0, 1, 2, 3 designate the four junctions of the network. The problem is to determine the current I flowing when a battery of potential difference E is connected between 0 and 1. By Ohm's law $E = IR$, where R designates the joint resistance of all branches between junctions 0 and 1.

A *star* of a network is defined as the branches meeting at a given junction. Let the branches of the network be regarded as independent generators of a Wang algebra. A *star element* of the algebra consists of the sum of the associated branches; thus the star element at junction 3 is $a + c + k$. The element N is defined to be the product in the Wang algebra of all the star elements except those at junctions 1 and 0. The element D is defined to be the Wang product of all the star elements except one. (It makes no difference which one.) The joint resistance R between 0 and 1 is symbolized by

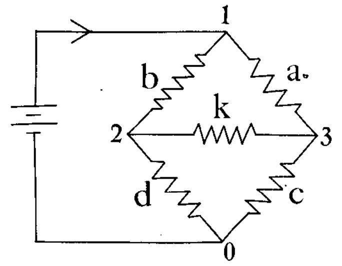

Fig. 1. A Wheatstone bridge.

$$R = \frac{N}{D}. \tag{2}$$

Here N and D are to be simplified by carrying out the indicated operations and making use of the Wang rules (1). After that the Wang algebra is dropped, and the resulting polynomials are considered ordinary polynomials, and the symbols $a, b, c, \cdots$, are taken to be the conductances of the corresponding branches.

In the network in Fig. 1 we have

$$\begin{aligned} N &= (b+d+k)(a+c+k) \\ &= ab + bc + bk + ad + cd + dk + ak + ck \end{aligned}$$

and

$$\begin{aligned} D &= (a+b)N \\ &= abc + acd + adk + ack + abd + bcd + bdk + bck. \end{aligned}$$

So

$$R = \frac{ab + bc + bk + ad + cd + dk + ak + ck}{abc + acd + adk + ack + abd + bcd + bdk + bck}.$$

That this is true may be seen by solving Kirchhoff's equations in the form

$$\begin{aligned} I &= (a+b)E & & -aE_3 \\ 0 &= \quad -bE + (b+d+k)E_2 & & -kE_3 \\ 0 &= \quad -aE + \qquad -kE_2 & & +(a+c+k)E_3. \end{aligned}$$

For a second example consider the graph in Fig. 2. Then

$$\begin{aligned} N &= (a+b+c)(c+d+e) \\ &= ac + ad + ac + bc + bd + be + cd + ce \end{aligned}$$

and

$$\begin{aligned} D &= aN \\ &= abc + abd + abe + acd + ace \end{aligned}$$

so

$$R = \frac{N}{D} = \frac{ac + ad + ae + bc + bd + be + cd + ce}{abc + abd + abe + acd + ace}.$$

Of course, the network in Fig. 2 is a series–parallel network, but notice that Wang algebra is simpler than

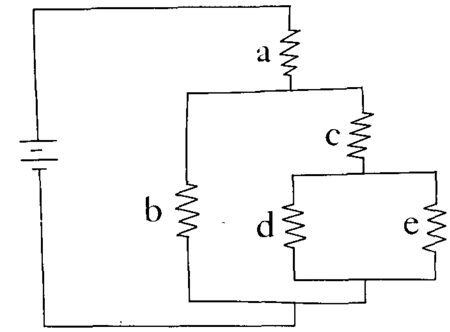

Fig. 2. A network.

rationalizing

$$\left(a^{-1}+\left(b+\left(c^{-1}+(d+e)^{-1}\right)^{-1}\right)^{-1}\right)^{-1}.$$

The Wang algebra may be used to evaluate any symmetric determinant. It is simply necessary to transform the determinant to network type. For example, to evaluate the three by three determinant:

$$\begin{vmatrix} e' & b & a \\ b & d' & k \\ a & k & c' \end{vmatrix}$$

let $-e=e'+a+b$, $-d=d'+b+k$, and $-c=c'+a+k$. Then the determinant is obtained by forming the Wang product

$$-S=(a+b+e)(b+k+d)(a+k+c).$$

[See J. J. Sylvester's unisignant determinant [6] or [8]. The Wang algebra can also be used with meshes instead of stars, see [3].

III. The Wang Algebra and Linear Independence

To make the definition of the Wang algebra more precise, we need to make use of vector spaces defined over the two element field. The two element field, sometimes called $GF(2)$, has two elements 0 and 1, with the rules $0+0=1+1=0$, $0+1=1+0=1$, and $0\cdot 1=1\cdot 0=0\cdot 0=0$, $1\cdot 1=1$. All notions of linear algebra, unless explicitly stated otherwise, carry over with minor modifications to vector spaces over the two element field. For instance vectors v_1,v_2,v_3 are linear dependent if there are $\lambda_1,\lambda_2,\lambda_3$ not all zero, $\lambda_1=0$ or 1, $\lambda_2=0$ or 1, etc., with $\lambda_1v_1+\lambda_2v_2+\lambda_3v_3=0$.

Let U be a finite dimensional vector space over the two element field. Let $\{e_i\}_{i=1}^m$ be a basis of U. Then the *Wang algebra*, $W(U)$, over U is the commutative algebra generated by $\{e_i\}$ subject to the rules:

1) $xy=yx$
2) $x+x=0$
3) $xx=0$.

For every set $I=\{i,j,\cdots,h\}$ of distinct integers from the set $\{1,2,3,\cdots,m\}$ set

$$E_I=e_i\cdot e_j,\cdots,e_h$$

then the E_I form a basis of $W(U)$, as I ranges over all subsets of $\{1,2,\cdots,m\}$. If $w\in W(U)$, then we may write

$$w=\sum_I d_IE_I \qquad (d_I=0 \text{ or } 1)$$

which we call the *canonical form* of w. If $d_I=1$ we call E_I a *term* of w.

For example, we compute

$$(e_1+e_2)(e_1+e_3)=e_1e_3+e_2e_3$$

so $E_{\{1,3\}}=e_1e_3$ and $E_{\{2,3\}}=e_2e_3$ are the terms of $(e_1+e_2)(e_1+e_3)$.

For the rest of this paper U and the basis $\{e_i\}$ are fixed. The elements $e_1,\cdots,e_m$ of this basis are called *edges*. Given any vector $u\in U$, we can write u uniquely as $u=\sum_{i\in I}e_i$ and thus we sometimes identify the vector u with the subset $I\subseteq\{1,2,3,\cdots,m\}$.

The following lemma shows how the Wang algebra may be used to determine the linear independence or linear dependence of a set of vectors of U. This lemma, as well as the one following it, may be found in [3].

Lemma 1

A set of vectors $p_1,p_2,\cdots,p_n$ of U are linearly independent if and only if $\Pi=p_1p_2\cdots p_n\neq 0$.

Proof: If $\{p_i\}$ are dependent, then by renumbering, we may assume that

$$P_n=\sum_{i=1}^{n-1}c_ip_i$$

so

$$\begin{aligned}\Pi&=(p_1\cdots p_{n-1})\cdot\sum_{i=1}^{n-1}c_ip_i\\&=\sum_{i=1}^{n-1}c_ip_1\cdot p_2\cdot\cdots p_{n-1}\cdot p_i.\end{aligned}$$

Each term on the right contains a repeated factor, and thus vanishes.

For the converse we proceed by induction. The case $n=1$ is clear. Therefore we assume that the lemma is true for integers less than n. Then writing p_n as a linear combination of the $\{e_i\}$ one of the coefficients must not vanish. By renumbering we may assume that the coefficient of e_1 is 1. Let us write

$$p_i'=p_i+c_ip_n, \qquad i=1,i,\cdots,n-1$$

where $c_i=0$ if the coefficient of e_1 in the expansion of p_i in terms of the e_i is 0, and $c_i=1$ if the coefficient is 1. Then p_i' does not contain e_1, i.e., the coefficient of e_1 in the expansion of p_i' is zero. If the $p_1,\cdots,p_n$ were linearly independent, then the vectors $p_1',\cdots,p_{n-1}'$ will also be linearly independent. Thus by induction we have $\Pi'=p_1',\cdots,p_{n-1}'\neq 0$. Then

$$\begin{aligned}\Pi'p_n&=(p_1+c_1p_n)(p_2+c_2p_n),\cdots,(p_{n-1}+c_{n-1}p_n)p_n\\&=p_1p_2,\cdots,p_n=\pi.\end{aligned}$$

This follows since $p_n \cdot p_n = 0$. Let $p_n = e_1 + q$, where the coefficient of e_1 in q is zero. Then

$$\Pi = \Pi' e_1 + \Pi' q$$

and clearly $e_1 \neq 0$. Moreover no term of $\pi' q$ can cancel with a term of $\Pi' e_1$. Thus $\Pi \neq 0$.

Lemma 2

Let V be a subspace of U, and suppose that p_i and q_i are two bases of V. Then

$$\prod_i p_i = \prod_i q_i \neq 0.$$

Proof: Let $p_i = \sum_{j=1}^r c_{ij} q_j$ for $i = 1,2,\cdots,r$. Then

$$\prod_i p_i = \prod_j \sum_j c_{ij} q_j = c \,\pi_j\, q_j$$

for some scalar $c \neq 0$. Therefore $c = 1$.

In view of the above lemma, if V is a subspace of U we set $\Pi(V) = \Pi_j p_j$ for some (and hence every) basis $\{p_j\}$ of V. $\Pi(V)$ is termed the *outer product of V*.

If V is a subspace of U, then we may represent V by giving a basis $\{p_j\}_{j=1}^r$ and then write each p_j as a linear combination of the fixed basis $\{e_i\}_{i=1}^m$. Thus we have $p_j = \sum_i e_i c_{ij}$, where all the c_{ij}'s are zero or one. Thus we can represent the subspace V in terms of a matrix $\{c_{ij}\}$. Note the column operations applied to the matrix $\{c_{ij}\}$ do not change the subspace that it represents. Moreover if we label the rows of $\{c_{ij}\}$ with the basis elements $\{e_i\}$, then we may interchange two rows of c_{ij} without changing V *as long as we also change the labels*. For example:

$$\begin{array}{c} e_1 \\ e_2 \\ e_2 \end{array}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array}\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \begin{array}{c} e_3 \\ e_1 \\ e_2 \end{array}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Certain types of bases of a subspace of U are important in network theory, for we shall see later that they correspond to trees.

Definition: A *diagonal basis* of a subspace V of U is a basis such that the representing matrix has the form:

$$\begin{array}{r} e_{h_1} \\ e_{h_2} \\ \cdot \\ e_{k_n} \\ \text{rest of } \{e_i\} \end{array}\begin{bmatrix} \\ I \\ \\ \\ A \end{bmatrix}$$

where I is the n by n identity matrix and A is arbitrary. Given such a basis, we say that it is a *diagonal basis with respect to the edges $e_{h_1},\cdots,e_{h_n}$*. The last basis in the last example is a diagonal basis with respect to e_3, e_1.

Lemma 3

Let Π be the outer product of the subspace V. Then $e_{h_1},\cdots,e_{h_n}$ is a term of Π in canonical form if and only if there is a diagonal basis of V with respect to $e_{h_1}\cdots e_{h_n}$.

Proof: See [3].

Let

$$x = \sum_{i=1}^m c_i e_i \quad \text{and} \quad y = \sum_{i=1}^m d_i e_i$$

be two vectors of U. Then we define an inner product by the formula

$$(x,y) = \sum_{i=1}^m c_i d_i \qquad (\text{mod } 2).$$

For example if $x = e_1 + e_3 + e_6$ and $y = e_2 + e_3$, then $(x,y) = 1$, $(x,x) = 1$ and $(y,y) = 0$. Thus we see that in general either $(x,y) = 1$ or $(x,y) = 0$. If $(x,y) = 0$, we say that x and y are orthogonal. This inner product is not positive definite ((x,x) may be zero when $x \neq 0$), but most of the usual properties of inner product spaces hold. In particular if we define $V' = \{y : (x,y) = 0 \text{ for all } x \in V\}$ then we have the following.

Lemma 4

If V is a subspace of U, then V' is also a subspace of U. In addition dim (V) + dim (V') = dim (U), *and* $(V')' = V$. *Here* dim (V) *denotes the number of elements in any (and hence every) basis of B.*

Proof: Similar to the usual proof.

If we are given a diagonal basis of V then we can give a diagonal basis of V' by a simple construction. Suppose the basis of V is given by

$$\begin{pmatrix} I \\ \hdashline A \end{pmatrix}$$

then a diagonal basis of V' is given by

$$\begin{pmatrix} A^T \\ \hdashline T \end{pmatrix}$$

where A^T denotes the transpose of A. We shall often write this as

$$\left[\begin{array}{c|c} V & V' \\ I & A^T \\ \hdashline A & I \end{array}\right].$$

We call the above the Brand tableau [2].

For instance, in the above example we obtain

$$\begin{array}{c} \\ e_3 \\ e_1 \\ e_2 \end{array}\begin{array}{c} \quad V \qquad V' \\ \left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right] \end{array}.$$

IV. Binary Matroids

In this section we give a (slightly nonstandard) definition of a binary matroid and relate the concept to graphs.

Definition: A pair (V, V') of a subspace V of U and its orthogonal complement V' is called a *binary matroid*. Recall that U has a fixed basis $\{e_i\}$, and the elements of this basis are called *edges*.

To see that our definition is equivilent to the usual definition see [5] or [9].

Consider the graph shown in Fig. 1.

Let U be the mod 2 vector space with preferred basis $\{a,b,c,d,e\}$. Thus $m=6$, $e_1=2$, $e_2=b,\cdots,e_5=e$.

Definition: A *cut* of a graph is a symmetric sum of stars. A *mesh* is a symmetric sum is simple closed cycles.

The cuts of the graph in Fig. 2 are

$\{a,b\}$
$\{a,c,k\}$
$\{d,e\}$
$\{b,d,k\}$
$\{b,c,k\}$
$\{a,c,k\}$
$\{a,b,d,e\}$.

The meshes are

$\{a,b,k\}$
$\{c,d,k\}$
$\{a,b,c,d\}$.

It can be seen that the set of all cut forms a subspace of U if we, for example identify $\{a,b\}$ with $a+b$, $\{a,c,d\}$ with $a+c+d$, etc. In a similar way the set of all meshes forms a subspace of U. Let us call the subspace of cuts the *cut space* and the subspace of all meshes the *mesh space*. These spaces are of use in network analysis, see [7].

Lemma 4

Let G be a graph. Then both the cuts and meshes form vector spaces over the two element field. If we set U to be the vector space over the two element field, with preferred basis being the branches of G, and set V to be the cut space, then the mesh space is equal to V'.

Proof: See [7].

A basis for the cut space of the graph in Fig. 2 is

$$\begin{array}{c} a\\ b\\ c\\ d\\ e \end{array}\begin{bmatrix} 1 & 1 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{c} b\\ c\\ e\\ a\\ d \end{array}\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 1 & 0\\ 0 & 1 & 1 \end{bmatrix}.$$

So the Brand tableau is

$$\begin{array}{c} \\ b\\ c\\ e\\ a\\ d \end{array}\begin{array}{c} \quad V \qquad\qquad V' \\ \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 1 & 1\\ 0 & 0 & 1 & 0 & 1\\ 1 & 1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0 & 1 \end{array}\right] \end{array}$$

and it can be seen that V' is the mesh space of the graph in Fig. 2.

In general, given a graph G, let V be its cut space, then V' will be its mesh space. Thus (V,V') will be a binary matroid, called the *matroid of G*.

In an arbitrary matroid, the elements of V are called *cuts* and the elements of V' are called *meshes*.

Given a graph G we could perform the same procedure as above, but instead take for V the mesh space of the graph, and thus V' will be the cut space of G. This is a special case of the following.

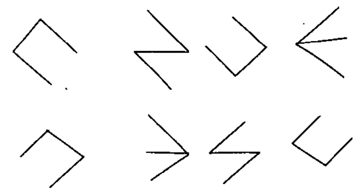

Fig. 3. The trees of G.

Theorem 1

Let (V,V') be a binary matroid. Then (V',V) is also a binary matroid, called the dual matroid of (V,V').

Proof: $(V')'=V$.

The above theorem illustrates the power to matroids over graphs. If a graph is planar then we can construct its dual (in the sense of Whitney, see [5]), and this will allow the principle of *electrical duality* to be applied. In fact if G is a planar graph then the matroid of the dual graph will be the dual matroid of G.

Not all binary matroids can arise as the matroid of a graph. For example let G be the complete graph on five nodes. In this graph there is a branch between every pair of the five nodes. Then the dual of the matroid of G cannot be the matroid of any graph. For a characterization of which matroids can or cannot arise as the matroid of a graph see [9].

V. Trees of Matroids

Let G be a connected graph. Recall that a *tree* of G is a subset T of the branches of G such that a) T contains no mesh, and b) T is maximal with respect to a).

Sometimes this notion is called a spanning tree. If G is the graph shown in Fig. 2 then $\{a,b,d\}$ is a tree while $\{a,b,c\}$ and $\{a,b\}$ are not.

Let us consider the Wang product of all but one star of the graph G:

$$\Pi=(a+b)(a+c+d)(d+e)$$
$$\times acd+ace+ade+bad+bae+bcd+bce+bde.$$

Notice that all the terms of Π correspond to trees of G. Note that now any term E_I, $I=\{j,k,\cdots,l\}$ corresponds with the set of edges $\{e_j,e_k,\cdots,e_l\}$. Thus we can associate with any term of an outer product a set of edges of the matroid. (Note: Previously we associated a set of edges with any vector $x\in U$. These two notions are not to be confused.) For example, *acd* corresponds to the tree $\{a,c,d\}$. Moreover all the trees of G correspond to terms of Π. This is a special case of the following.

Lemma 5

Let G be a connected graph. Then the trees of G (Fig. 3) are exactly the sets of edges corresponding to the terms in the outer product of the cut space.

Proof: See [3].

The above lemma leads us to the following definition.

Definition: Let (V,V') be a binary matroid. A *tree* of (V,V') is a subset of the preferred basis corresponding to a term of the outer product of V. A cotree is a subset corresponding to a term of the outer product of V'.

We see by definition that a tree of (V,V') is a cotree of the dual binary matroid, and that a cotree of a binary matroid is a tree of the dual binary matroid.

Theorem 2

$T=e_{h_1},\cdots,e_{h_r}$ is a tree if and only if there is a basis of the form

$$\begin{array}{r}e_{h_1}\\ e_{h_2}\\ \vdots\\ e_{h_r}\\ \text{rest of } \{e_i\}\end{array}\left[\begin{array}{c} \\ \\ I\\ \\ \hline A\end{array}\right]$$

where I is the r by r identity matrix and A is arbitrary.

Proof: See Lemma 3.

Corollary: All trees have the same cardinality, namely the dimension of V.

By duality, the above statements hold for cotrees, i.e., a cotree corresponds to a diagonal basis of V', and they all have the same cardinality, namely dim (V'). Thus in binary matroids, we get two theorems for the price of one.

Theorem 3

The complement of a tree is a cotree.

Proof:

$$\begin{array}{lcc} & V & V' \\ \text{tree } \{e_1,\cdots,e_r\} & \multicolumn{2}{c}{\left(\begin{array}{c|c} I & A^T \\ \hline A & I \end{array}\right)} \\ \text{cotree } \{e_{r+1},\cdots,e_n\} & & \end{array} \qquad (*)$$

Theorem 3′

The complement of a cotree is a tree.

Theorem 4

A tree contains no mesh.

Proof: If $\{e_1,\cdots,e_r\}$ is a tree then there is a diagonal basis with respect to $e_1,\cdots,e_r$. Therefore the situation is as indicated (*). It can be seen from the Brand tableau that every mesh (element of V') contains a member of $\{e_{r+1},\cdots,e_n\}$.

Theorem 4′

A cotree contains no cut.

Theorem 4′ is obtained from Theorem 4 by replacing V by V'.

We are now in a position to show that the trees of a binary matroid can be defined in the same terms as the trees of a graph, that is that our definition of trees in a binary matroid is equivalent to the graphical definition given in a) and b). But first we need the following theorem.

Theorem 5

Let T be a tree and let x be an edge that is not in T, then $T\cup x$ contains a mesh.

Proof: Without loss of generality $x=e_{r+1}$ in the preceding Brand tableau (*). Then $T\cup x$ contains the mesh corresponding to the first V' column.

Theorem 6

T is a tree if and only if T contains no mesh and is maximal with respect to that property.

Proof: One direction follows from Theorems 4 and 5. For the converse direction we need a lemma.

Lemma 6

Suppose that F contains no mesh. Let a basis of V be given as

$$\begin{array}{r} F \\ \{e_i\}-F \end{array}\begin{pmatrix} A \\ B \end{pmatrix}$$

then the rows of F are linearly independent.

Proof: If the rows of A were not linearly independent, then one could find a nonzero solution to $A^Tx=0$, where x is a zero-one vector. It is easily seen that the vector $\begin{bmatrix} x \\ 0 \end{bmatrix}$ is orthogonal to every cut and is thus a mesh. Moreover this mesh is contained in F.

Proof of Theorem 6 (continued): If T is a maximal set of edges containing no mesh then by column operations we may diagonalize the matrix A in the above lemma.

VI. Separation in Matroids

In analogy with graphs we make the following definition.

Definition: A binary matroid (V,V') is *separable* if there is a partition of the set of edges $\{e_i\}=E'\cup E''$ where $E'\cap E''=\varnothing$ and $E'\neq\varnothing$, $E''\neq\varnothing$, and a basis of V of the form

$$\begin{array}{r} E' \\ E'' \end{array}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where 0 denotes an appropriately sized zero matrix, and A and B are arbitrary. (V,V') is called *nonseparable* if it is not separable.

Lemma 7

A binary matroid is separable if and only if there is a diagonal basis of the form

$$\left[\begin{array}{cc} \multicolumn{2}{c}{I} \\ \hline C & 0 \\ 0 & D \end{array}\right]$$

where C and D are arbitrary, 0 is the zero matrix of the appropriate order, and I is the identity matrix.

Proof: Apply column operations (Gaussian elimination) to the matrix in the definition of separable.

Theorem 7

A binary matroid is separable if and only if its dual is separable.

Proof: Applying the brand tableau to the matrix of Lemma 7 we obtain

$$\begin{array}{cc} V & V' \\ \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{I} & C^T & 0 \\ & & 0 & D^T \\ \hline C & 0 & \multicolumn{2}{c}{I} \\ 0 & D & & \end{array}\right] \end{array}.$$

The theorem follows.

The Wang algebra, however, affords an algebraic way of telling whether or not a binary matroid is separable or not. This is useful because of the above definition is difficult to check in practice.

Definition: Let $\pi=\Sigma_I d_I E_I$ be the outer product of a subspace V in canonical form. We define the *Bott–Duffin discriminant*, D_v of the subspace V of U as the multilinear form

$$D_V(x_1,\cdots,x_n)=\Sigma d_I x_{I_1}\cdots x_{I_r}.$$

Many properties of the Bott–Duffin discriminant were studied in [1].

If (V,V') is separable then we have a basis of V of the form

$$\begin{array}{cc} V_1 & V_2 \\ \left[\begin{array}{c|c} A & 0 \\ 0 & B \end{array}\right] \end{array}.$$

Let $D_1=D_{V_1}$, and $D_2=D_{V_2}$. It can now be seen that $D_V=D_1D_2$, and thus the Bott–Duffin discriminant of V is *reducible*, that is it is the product of two nonconstant polynomials. This in fact characterizes separable binary matroids.

Theorem 8

The Bott–Duffin discriminant is reducible if and only if the matroid is separable.

Proof: We have just proved one direction. For the converse suppose that $D_V=D_1D_2$. Setting all the variables except x_i to algebraically independent transcendentals we have

$$ax_i+b=D_1(\cdots,x_i,\cdots)D_2(\cdots,x_i,\cdots)$$

for some real numbers a and b. Thus the two polynomials D_1 and D_2 contain distinct variables. It now follows that each of the D_j's is multilinear and by a suitable normalization we may assume that all nonzero coefficients of D_1 and D_2 are equal to one. Picking some tree of (V,V') we have a basis of V of the form

$$\begin{array}{c} T\cap F_1 \\ T\cap F_2 \\ \bar{T}\cap F_1 \\ \bar{T}\cap F_2 \end{array}\begin{bmatrix} I & 0 \\ 0 & I \\ A & B \\ C & D \end{bmatrix}$$

where F_1 corresponds to the variable contained in D_1, F_2 corresponds to the variable contained in D_2, T is a tree of (V,V'), and $\bar{T}$ denotes the complement of T. The conclusion of the theorem is equivalent to $B=0$, $C=0$. Suppose (say) $C\neq 0$. Then there will be a tree of V containing $x\in\bar{T}\cap F_2$; all but one element of $T\cap F$, and all the elements of $T\cap F_2$. This tree must be a term of $D=D_1D_2$, but this is impossible.

VII. Regular Matroids

Suppose that instead of U we take $\mathbb{R}^N$ with basis $\{e_i\}$. Then when will the Wang trick work?

Definition: Let S be a subspace of $\mathbb{R}^N$. Then the *Bott–Duffin discriminant* of S is defined by

$$D(x_1,\cdots,x_N)=\det(GP+P')$$

where G is a diagonal matrix of the variables x_i, P is the perpendicular projection onto S, and P' is the perpendicular projection onto $S^\perp$.

Theorem 8

Let S be an m dimensional subspace of $\mathbb{R}^N$. Let f be the Bott–Duffin discriminant of S. Let V be the set of vectors of S whose components are $+1$, -1, or 0. Then $f=c\Pi$ for some scalar c and some Wang product Π if and only if V forms an m dimensional vector space under addition modulo 2.

Proof: See [1].

Definition: A vector $x\in\mathbb{R}^N$ is *just* if all of its components are $+1$, -1, or 0.

Definition: A *regular matroid* is a pair $(S,S^\perp)$ of a subspace of $\mathbb{R}^N$ and its orthogonal complement, such that S satisfies the conditions of Theorem 8.

Proofs of the following are contained in [1] or [3].

Proposition 1: If $(S,S^\perp)$ is a regular matroid, then so is $(S^\perp,S)$.

Proposition 2: Let V be as in Theorem 8, and let W be the just vectors of S; then $W=V'$. In particular (V,W) is a binary matroid.

By Theorem 8 and the above proposition we can compute discriminants of a regular matroid by applying the Wang trick to (V,W). The following theorem from [1] tells whether or not a binary matroid comes from a regular matroid.

Theorem 9

Let (V,V') be a binary matroid, then (V,V') is the binary matroid of a regular if and only if ψ_{ij} is a perfect square for each i and j. Here

$$\psi_{ij} = -\frac{\partial^2}{\partial x_i \partial x_j} \log \det (D_V).$$

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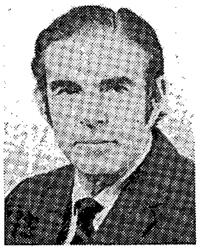

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Fenchel Duality of Nonlinear Networks

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***Abstract*—This paper contains an analysis of the interconnection of nonlinear n-port networks. The primary goal of the paper is to extend to nonlinear networks the results derived previously for linear networks. The class of nonlinear networks considered have impedance functions that are subdifferentials of convex functions. Using the properties of the convex and impedance functions, it is shown that the network connections induce a natural operation on the class of impedance functions. The classical duality of current and voltage is expressed by using the concepts of conjugate functions, also known as Fenchel duality. Inequalities relating different network connections are also presented.**

I. INTRODUCTION

IN THIS PAPER we consider the interconnection of nonlinear n-port networks. Our primary goal is to extend to nonlinear networks the results derived in [1] for linear networks. The networks are represented by their impedance functions; their internal structure is irrelevant to our discussion. We consider a special class of nonlinear networks whose impedance functions are the subdifferentials of convex functions. By using the properties of both the convex and impedance functions, and by using the formulation of Kirchhoff's laws given in [1], we are able to show that our results for linear networks extend to nonlinear networks. For example, we are able to show that a network connection induces an operation on the class of impedance functions; this operation is called the router sum, and represents the impedance function of the interconnected network. We also show that the natural duality of the current and voltage variables may be expressed by using the concepts of conjugate functions, also known as Fenchel duality [8].

In the case of linear networks, n-ports are represented by positive semidefinite impedance matrices. Then, by using only Kirchhoff's laws, we have shown that any

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