Topological Entanglement and Quantum Entanglement

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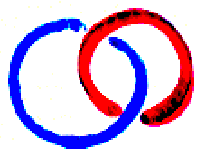
I.

L 

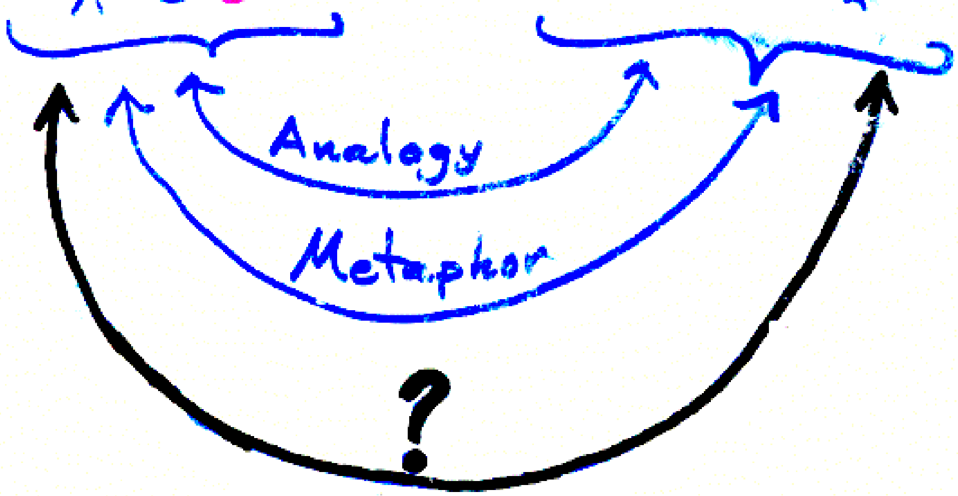
A topological linking (entanglement)

$$\psi = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

A quantum entanglement

$L \neq O\,O$

$\psi \neq \psi_1 \otimes \psi_2$



Quantum

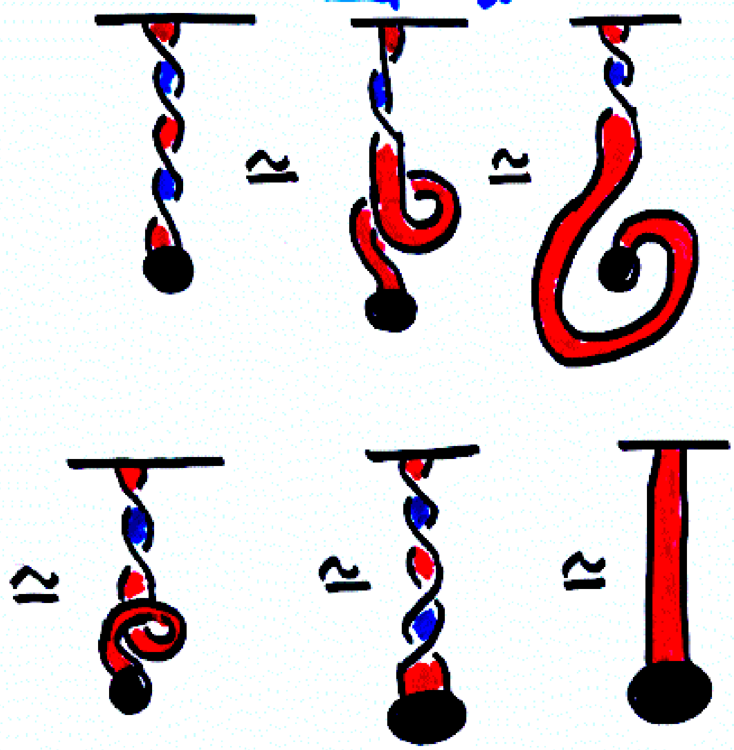
$$\psi = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

"Red Observer"

"Blue Observer"

Quantum Nonlocality

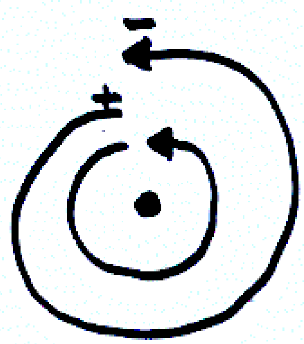
Topology



Dirac String Trick

$\longleftrightarrow$ 2-fold cover

$$SU(2) \xrightarrow{2-1} SO(3)$$



$\longleftrightarrow$ Change in phase of wavefunction for a fermion.

Example of **P.K. Aravind** ②

(In "Potentiality, Entanglement and Passion-at-a-Distance" Kluwer (1997) ed. by R.S. Cohen et al)

Borrowmean Rings and the **GHZ state**

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\beta_1\rangle|\beta_2\rangle|\beta_3\rangle - |\alpha_1\rangle|\alpha_2\rangle|\alpha_3\rangle\right)$$

- 3 particles
- all spins in z-direction.

Measure any particle → state becomes disentangled.

Borromean Rings

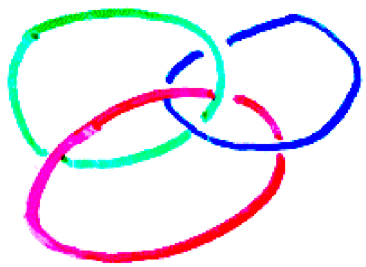
But if you change basis

$$|\psi\rangle = \frac{|\beta_{1x}\rangle}{\sqrt{2}}\left(\frac{|\beta_2\rangle|\beta_3\rangle - |\alpha_2\rangle|\alpha_3\rangle}{\sqrt{2}}\right)$$

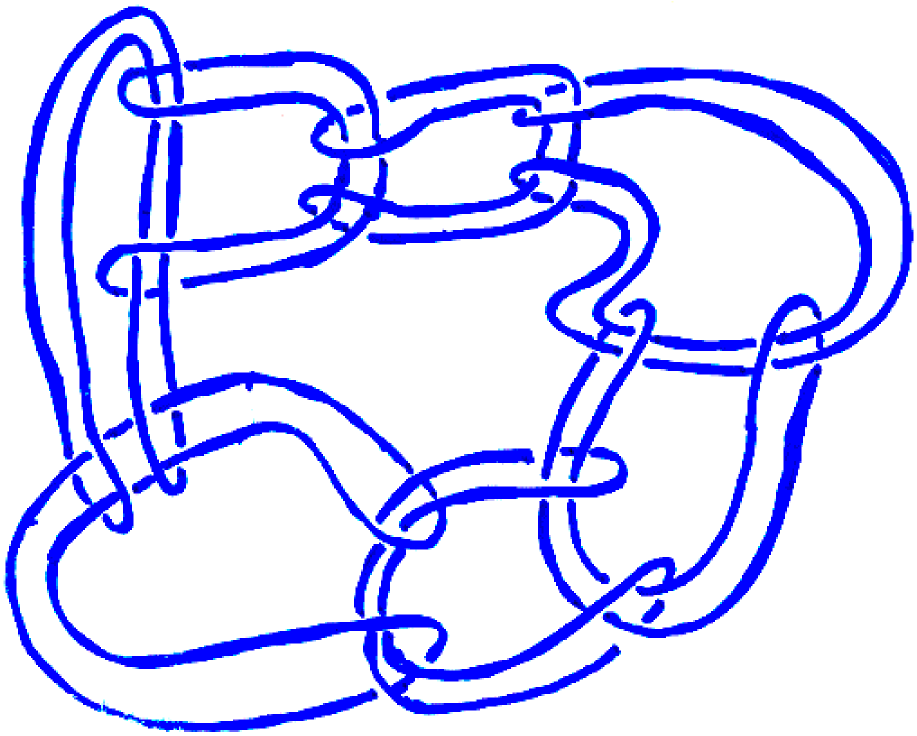
$$+ \frac{|\alpha_{1x}\rangle}{\sqrt{2}}\left(\frac{|\beta_2\rangle|\beta_3\rangle + |\alpha_2\rangle|\alpha_3\rangle}{\sqrt{2}}\right)$$

where $|\beta_{1x}\rangle$ & $|\alpha_{1x}\rangle$ denote spin-up & spin-down states of particle 1 in x direction.

And analogy is now to



each pair is linked once!



$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\beta_1\rangle \cdots |\beta_N\rangle - |\alpha_1\rangle \cdots |\alpha_N\rangle\right)$$

<u>Question</u>: Investigate the <u>collection</u> of linking patterns that describe the entanglement of a <u>given state</u>.

II. Operator View Point

$$\mathcal{C} = CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$|x\rangle \longrightarrow \bullet \longrightarrow |x\rangle$$
$$|y\rangle \longrightarrow \oplus \longrightarrow |y \oplus x\rangle$$

unitary entangling operator

$$\left.\begin{aligned} \mathcal{C}|00\rangle &= |00\rangle \\ \mathcal{C}|01\rangle &= |01\rangle \\ \mathcal{C}|10\rangle &= |11\rangle \\ \mathcal{C}|11\rangle &= |10\rangle \end{aligned}\right\}$$

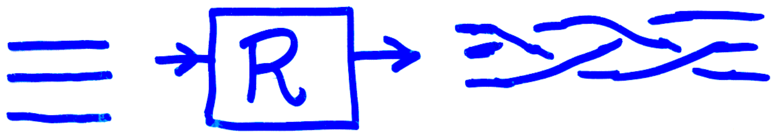
$$\psi \rightarrow \boxed{\mathcal{C}} \rightarrow \psi'$$

unentangled

entangled

$$\mathcal{C}\left(\left[a|0\rangle + b|1\rangle\right] \otimes |0\rangle\right)$$
$$\|$$
$$a|00\rangle + b|11\rangle$$

entangled

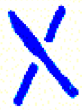


Needed: A quantum weaving machine

On topological side

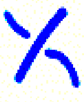
R can be an elementary braid.

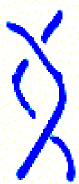
So: How to associate a unitary operator to an elementary braid?

 $\rightsquigarrow R$

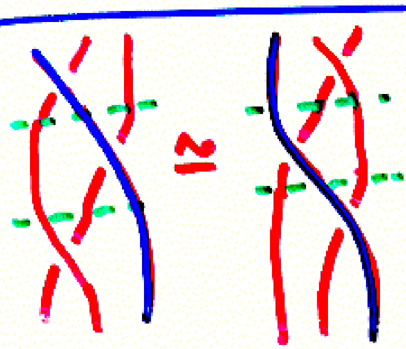
 $\rightsquigarrow R \otimes I$

 $\rightsquigarrow I \otimes R$ etc.

 $\rightsquigarrow R \Rightarrow$  $\rightsquigarrow R'$

and  $\simeq\ )(\ \Rightarrow R' = R^{*} = R^{-1}$

(since R is unitary)



$(R\otimes I)(I\otimes R)(R\otimes I)$
$\|$
$(I\otimes R)(R\otimes I)(I\otimes R)$

The Yang-Baxter Equation

An Example

$$R = \begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & & & & \\ 01 & & & & \\ 10 & & & & \\ 11 & & & & \end{array} \quad
\begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array}\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 0 & b \end{bmatrix}$$

a, b, c, d unit complex nos. (so $\bar{a} = a^{-1}$ ...)

$$R|00\rangle = a|00\rangle$$
$$R|01\rangle = c|10\rangle$$
$$R|10\rangle = d|01\rangle$$
$$R|11\rangle = b|11\rangle$$

$\mathrm{Det}(R) = -abcd$

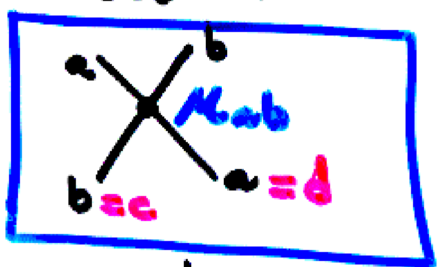
(So $-abcd = 1$ if want $\mathrm{Det}(R) = 1$.)

Claim: • R is unitary
• R satisfies the Yang-Baxter Equation

More generally, let

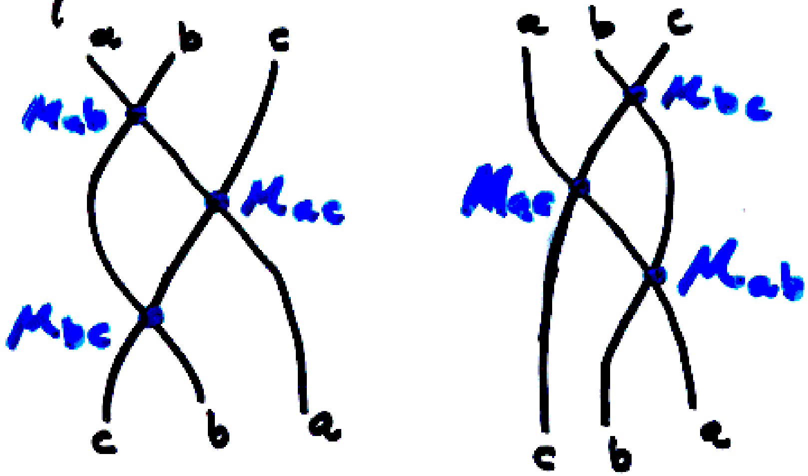
$M=(M_{ab})$ be any $n \times n$ matrix whose entries $M_{ab} \in \mathbb{C}$ with $|M_{ab}|=1$.

Let $R^{ab}_{cd} = M_{ab}\,\delta^a_d\,\delta^b_c$



$$(R^*)^{ab}_{cd} = \overline{M}_{ba}\,\delta^a_d\,\delta^b_c \Rightarrow R^* = R^{-1}$$

and R satisfies the Yang-Baxter Equation.



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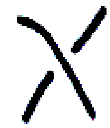
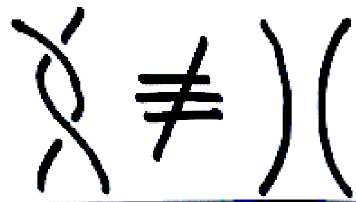
$$R = \begin{pmatrix} a & & & \\ & & c & \\ & d & & \\ & & & b \end{pmatrix} \leftrightarrow$$ 

$$R^* = \begin{pmatrix} \bar{a} & & & \\ & & \bar{d} & \\ & \bar{c} & & \\ & & & \bar{b} \end{pmatrix} \leftrightarrow$$ 

$$R^2 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix} = \begin{pmatrix} a^2 & & & \\ & cd & & \\ & & dc & \\ & & & b^2 \end{pmatrix}$$

$R^2 \neq I$



<u>Lemma</u>. Let $\psi = |0\rangle + |1\rangle$
$\phi = R(\psi \otimes \psi)$. Then

$\boxed{\phi \text{ is entangled if } ab \neq cd.}$

<u>Pf</u>. $R(\psi \otimes \psi) = R[|00\rangle + |01\rangle + |10\rangle + |11\rangle]$
$= a|00\rangle + c|10\rangle + d|01\rangle + b|11\rangle$

$(x|0\rangle + y|1\rangle) \otimes (z|0\rangle + w|1\rangle)$
$= \underline{xz}|00\rangle + \underline{yz}|10\rangle + \underline{xw}|01\rangle + \underline{yw}|11\rangle$

Special Case

$$R = \begin{pmatrix} a & & & \\ & & c & \\ & c & & \\ & & & a \end{pmatrix}.$$

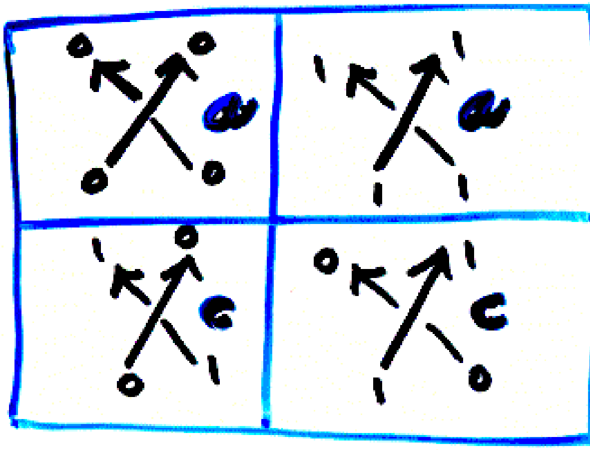
Then R entangles $(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)$ if $a^2 \neq c^2$. Hence entangles if $c^2/a^2 \neq 1$.

Here $R^2 = \begin{pmatrix} a^2 & & & \\ & c^2 & & \\ & & c^2 & \\ & & & a^2 \end{pmatrix}$

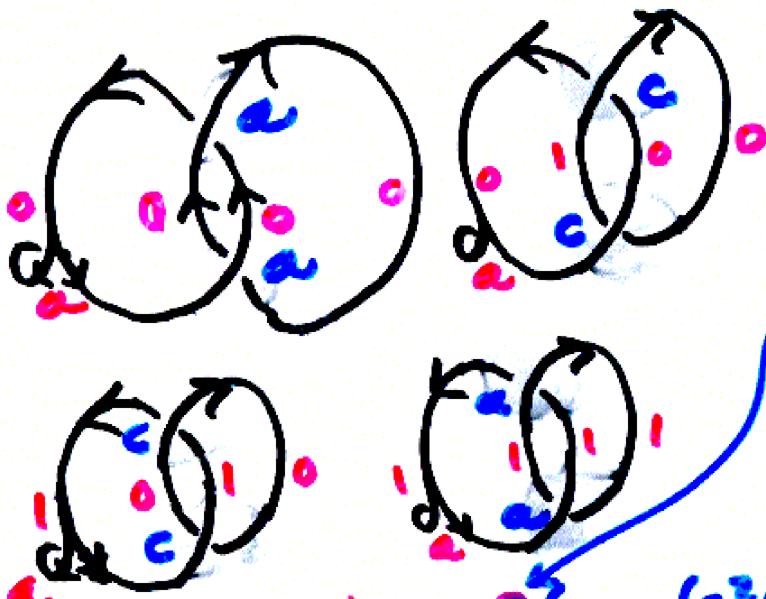
Thus $a^2 = c^2 \Rightarrow R^2 = a^2 \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

and  and)(only differ by global phase.

So here the entanglement condition and the ability to detect linking are coincident.



$$R = \begin{pmatrix} a & & & \\ & & c & \\ & c & & \\ & & & a \end{pmatrix}$$

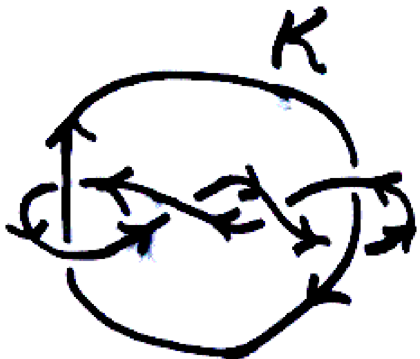


Here $3 = w(K)$

$$a(2a^2 + 2c^2) = 2a^3\left(1 + (c^2/a^2)\right)$$

Let $Z_K = \dfrac{\text{sum of contrib}}{2a^{w(K)}}$

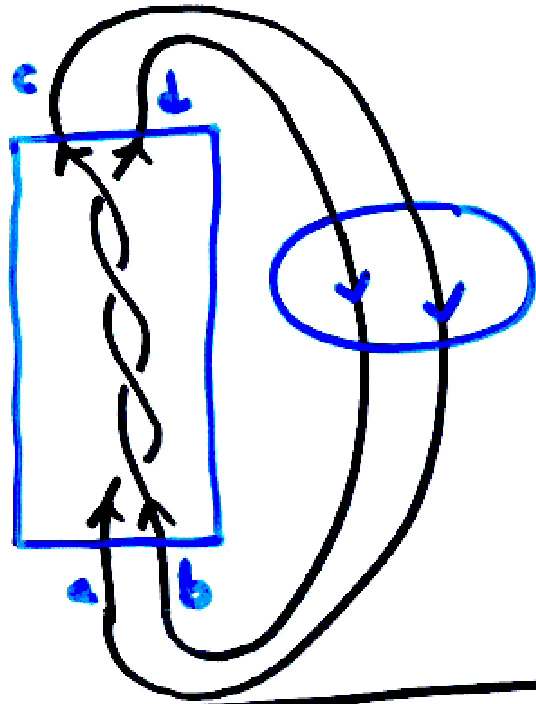
$$\Rightarrow \boxed{Z_K = 1 + (c^2/a^2)^{\text{linking\#}(K)}}$$



$lk(K) = 2$

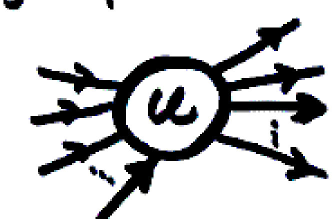
$$Z_K = 1 + (c^2/a^2)^2$$

Here we have used the solution to the Yang-Baxter equation to create an **invariant of links Z_K** that is obtained by summing over all values of quantum states compatible with the structure of the link. In braid form we can see that this involves a form of "quantum feedback"



$\Rightarrow d=b$
$c=a$

More generally one can consider a network $\mathcal{N}$ of unitary operators



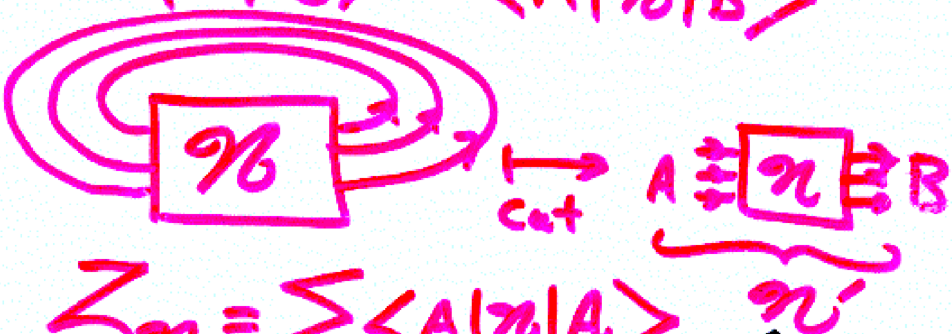
and define $Z_{\mathcal{N}} = \sum_{\substack{\text{quantum} \\ \text{states of } \mathcal{N}}} \prod \mathcal{U}^{ab\cdots}_{cd\cdots}$

"Network Amplitude"

Such a sum is analogous to a path integral.

<u>Problem</u>. Find good quantum computing algorithms for obtaining $Z_{\mathcal{K}}$.

<u>Discussion</u>. Cutting $\mathcal{K}$ will allow $\mathcal{K}$ to be configured as $\langle A | B \rangle = \langle A | \mathcal{K} | B \rangle$



$$Z_{\mathcal{K}} = \sum_{A} \langle A | \mathcal{K} | A \rangle$$

In general need exponentially many runs of $\mathcal{K}'$ to determine $Z_{\mathcal{K}}$.

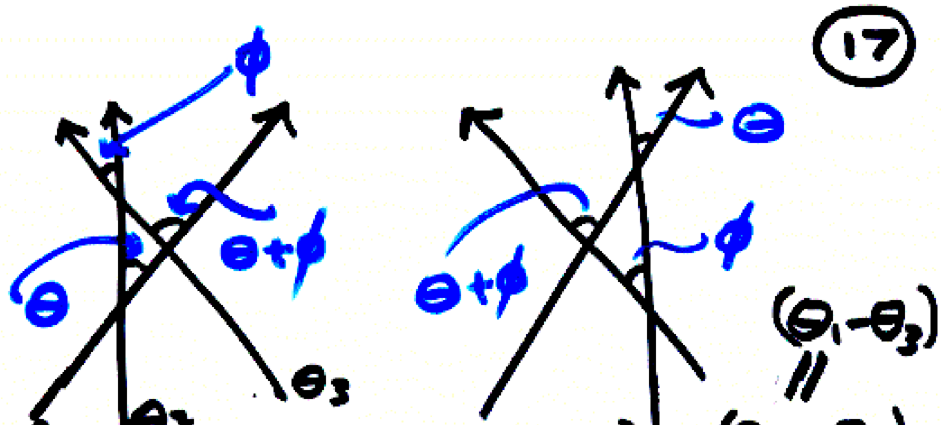
Such Networks have different degrees of physical interpretation. Here is one general form of model used in knot theory:

$\lambda^{a\theta/2\pi}$

$$S^{ab}_{cd}(\theta) = R^{ab}_{cd}\,\lambda^{\frac{\theta(d-a)}{2\pi}}$$

Here R^{ab}_{cd} is a solution of the Yang–Baxter equation s.t. $a+b = c+d$ when $R^{ab}_{cd} \neq 0$.

- R^{ab}_{cd} is "spin preserving"
- θ is a rapidity parameter $\leftrightarrow$ (momentum difference)



$\theta = \theta_1 - \theta_2$: $(\theta_1-\theta_3) = (\theta_1-\theta_2)+(\theta_2-\theta_3)$

Relativistic Context

$$E^2 = c^2p^2 + m^2c^4$$

Let $c=m=1$. Then

$$E^2 = p^2 + 1$$

$$E = \cosh(\theta) = \frac{e^{\theta}+e^{-\theta}}{2}, \quad p = \sinh(\theta) = \frac{e^{\theta}-e^{-\theta}}{2}$$

$[E,p] \longleftrightarrow [e^{\theta}, e^{-\theta}]$ in light-cone coords.

$\Rightarrow$ rapidity difference

$\underline{\underline{\theta_1 - \theta_2}}$ $\underline{\underline{\text{Lorentz Invar}}}$

<u>Key Question</u>: What is the role of quantum entanglement in quantum computation?

We have seen that computations related to <u>topological linking</u> can also be related to quantum entanglement. On the other hand, some subtle computations do not involve entanglement.

For example, we can set up a one-qubit computation that gets information on the Jones representation for 3-strand braids.

σ_1

σ_2

σ_1, σ_2 are each represented by 2×2 matrices.

$$U_1 = \begin{bmatrix} \delta & 0 \\ 0 & 0 \end{bmatrix} = \delta \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$U_2 = \begin{bmatrix} \delta^{-1} & \sqrt{1-\delta^{-2}} \\ \sqrt{1-\delta^{-2}} & \delta - \delta^{-1} \end{bmatrix} = \delta \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} a & b \end{pmatrix}$$

$$\left.\begin{aligned} U_1^2 &= \delta U_1 \\ U_2^2 &= \delta U_2 \\ U_1 U_2 U_1 &= U_1 \\ U_2 U_1 U_2 &= U_2 \end{aligned}\right\} \text{Temperley Lieb Algebra}$$

$$\boxed{\begin{aligned} a^2 + b^2 &= 1 \\ a &= \delta^{-1} \\ b &= \sqrt{1-\delta^{-2}} \end{aligned}}$$

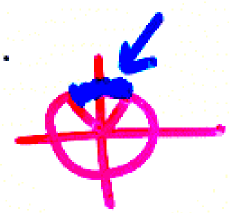
$$\left.\begin{aligned} \sigma_1 &\longmapsto A + A^{-1} U_1 \\ \sigma_2 &\longmapsto A^{-1} + A U_2 \end{aligned}\right\} \underline{\underline{\text{unitary}}}$$

Want $A \in$ unit circle in cx plane

$\delta = -A^2 - A^{-2}$

$\sqrt{1-\delta^{-2}}$ real

$\Rightarrow 2\,\mathrm{Re}(A^4) + 1 \geq 0$



We mention these representations because the existence of unitary braid representations such as those, derived from projectors, is somehow at the heart of the relationship between topology and the combinatorial formalism of quantum theory.

Bare Dirac formalism:

$$u = ><$$

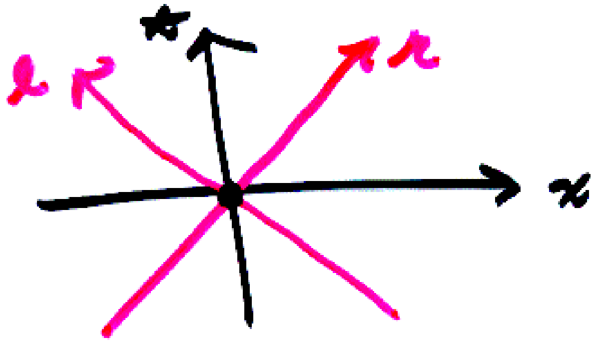
$$u^2 = ><>< = \delta >< = \delta u.$$

$$V =][\, , \quad V^2 =][][= \mu V.$$

$$\Rightarrow u V u = > \underbrace{<] [>} < = \triangleleft\triangleright\, u$$

$$V u V =] \underbrace{[> <]} [= \triangleleft\triangleright\, V.$$

Another Network Summation



$r = \dfrac{(t+x)}{2}$

$l = \left(\dfrac{t-x}{2}\right)$

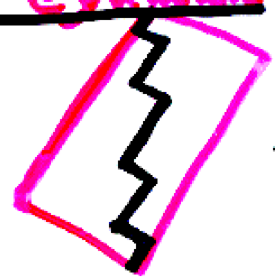
Light Cone Coords

1+1 Dirac Equation

$$\left.\begin{aligned} \frac{\partial \psi_R}{\partial r} &= i\psi_L \\ \frac{\partial \psi_L}{\partial l} &= i\psi_R \end{aligned}\right\} \psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$$

$$\updownarrow \qquad E = c\alpha p + \beta mc^2, \quad i\hbar\frac{\partial \psi}{\partial t} = c\sqrt{p^2 + m^2c^2}\,\psi$$

Feynman Checkerboard

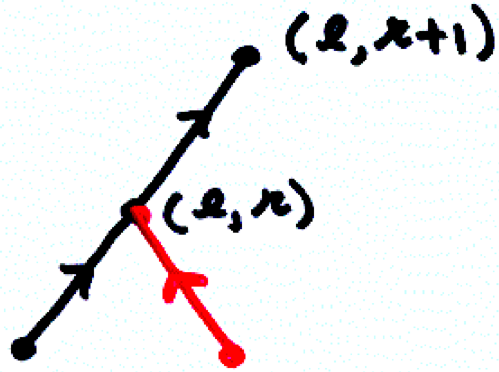


$$\sum_{\substack{\text{Paths}\\ P}} i^{\#\text{corners}(P)}$$

$$\sum_{\text{states}} |LRLR\cdots\rangle\, i^{\#(LR \text{ or } RL)}$$

Discrete Dirac

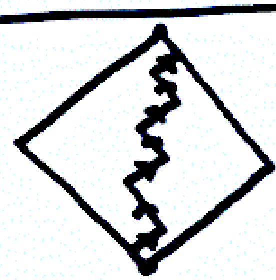
21.1



$$\psi_R(l, r+1) = \psi_R(l, r) + i\psi_L(l, r)$$

$$\psi_R(l, r+1) - \psi_R(l, r) = i\psi_L(l, r)$$

$$\frac{\partial \psi_R}{\partial r} = i\,\psi_L$$



Some Fundamental Questions

If we take Dirac's dictum

$$\{\,,\,\} \rightsquigarrow [\,,\,]$$

Poisson Brackets — Commutator Brackets

Classical — **Quantum**

to heart, what is the nature of this transition?

$$i\hbar\frac{\partial\psi}{\partial t} = E\psi$$

Can E be a function of the quantum state itself (Sam L's form of feedback) or must it always be described through a transition from classical realm as in

$$x \longmapsto x$$
$$p \longmapsto \frac{\hbar}{i}\frac{\partial}{\partial x} \quad ?$$

Are there discrete models underlying quantum phenomena? If so, how does this affect limits of quantum computation?

This is another reason for paying attention to solutions to Yang-Baxter equation, because they arise in the course of "quantizing" Lie algebras and the topology seems wedded to certain aspects of physics.

Can we classify quantum entanglements in terms of braiding operators?

Classify dynamics generated by braiding operators.

Classify unitary solutions to Yang-Baxter equation.