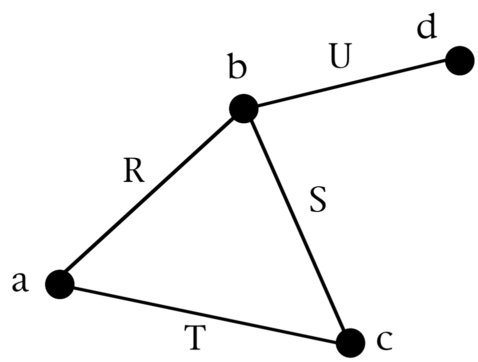
Euler's Formula for Plane Graphs

by LK

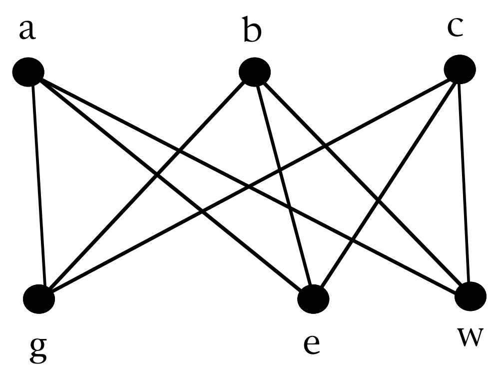
Leonhard Paul Euler (1707 - 1783) was a pioneering Swiss mathematician who spent most of his life in Russia and Germany. See <http://en.wikipedia.org/wiki/Leonhard_Euler> for information about his life and works.

This note is devoted to a statement of a formula first discovered by Euler through his study of polyhedral geometry. We will state the formula in terms of plane graphs. First, recall that a *graph* is a mathematical structure consisting in two sets, a set of *nodes* (or vertices) V and a set of *edges* E. Each edge is associated with one or two nodes. More than one edge can be associated with the same nodes. If R is an edge of G associated with the nodes a and b, we will write [R] = {a,b}. We call [R] the *boundary* of R.

We represent graphs by making diagrams where the nodes are points (or blobs) and the edges are curves whose ends meet the associated nodes. For example, the following drawing represents the graph G with V = {a,b,c,d} and E = { R,S,T,U} where[R] = {a,b}, [S] = {b,c}, [T] = {c,a} and [U] = {b,d}.



We say that a graph is planar if it can be represented in this way so that it is drawn in the plane and no edge touches any other edge except at its endpoints. Thus the graph shown above is indeed planar. In representing graphs we do not insist that the edges be straight. We can also represent graphs in the plane that are in fact not planar by allowing some edges to cross one another. Here is an example of a non-planar graph that is so drawn in the plane.



As you can see, in this graph each of the nodes a,b,c is connected by an edge to each of the nodes g,e,w. It turns out that there is no way to accomplish this and draw the graph in the plane without any extra crossings of edges. Here I have drawn it with 9 extra crossings. This is far too many! What is the least number of such crossings that are needed to represent this graph in the plane?

One more concept: A graph G is said to be *connected* if it is possible to make a path from any node to any other node by walking along edges of the graph. The two graphs we have illustrated so far are both connected. We leave it to you to give an example of a graph that is not connected. Ok?

Now go back to the first graph that we drew. It is planar. There are no extra crossings of edges. And you can see that the plane is divided up into two regions by this graph, an outer unbounded region and the inner region of the triangle formed by the nodes a,b,c and the edges R,S,T. Let
v = the number of nodes of G,
e = the number of edges of G,
f = the number of regions made by G in the plane.
Then here we have
v = 4, e = 4, f = 2.

Theorem (Euler). Let G be any connected plane graph (i.e. G is represented in the plane with no extra crossings between its edges). Let v,e,f denote the number of nodes, edges and faces of G in the plane. Then **v - e + f = 2.**

Illustrate Euler's formula with other examples of plane graphs, and find a proof!