Exam2 - Math 215 - Fall 2010

Write all your proofs with care, using full sentences and correct reasoning.

1. Prove $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$ for all $n = 1, 2, 3, \cdots$.

Solution. This is a straightforward induction argument. Solution omitted.

2. (a) Prove that there is a $1-1$ correspondence between the set

$$O = \{1, 3, 5, 7, 9, 11, \cdots\}$$

of odd natural numbers and the set N of all natural numbers.

Solution. $n \longrightarrow 2n - 1$.

(b) Let

$$S = \{1, 4, 9, 16, 25, 36, 49, \cdots\}$$

be the set of natural numbers that are squares. Show that there is a $1-1$ correspondence between S and the set N of all natural numbers.

Solution. $n \longrightarrow n^2$.

3. Prove that the single statement B is equivalent to:

$$(A \vee B) \wedge (A \Rightarrow B).$$

In your proof, do *not* use truth tables. Use the fact that

$$A \Rightarrow B \ = (\sim A) \vee B,$$

and give a completely algebraic proof.

Solution.

$$(A \vee B) \wedge (A \Rightarrow B) = (A \vee B) \wedge (\sim A \vee B)$$
$$= (A \wedge \sim A) \vee B = F \vee B = B.$$

4. Define the composition of the function $f : X \longrightarrow Y$ and the function $g : Y \longrightarrow Z$ to be the function $g \circ f : X \longrightarrow Z$ with $g \circ f(x) = g(f(x))$ for all $x \in X$. Prove that if f is injective and g is injective, then $g \circ f$ is injective.

Solution. Suppose that $g \circ f(a) = g \circ f(b)$. Then $g(f(a)) = g(f(b)$, so we have $f(a) = f(b)$ since g is injective. But $f(a) = f(b)$ implies that $a = b$ since f is injective. This proves that $g \circ f$ is injective.

5. Given sets A and B, consider the following two statements about a function $f : A \longrightarrow B$.
(i) $\forall b \in B, \exists a \in A$ such that $f(a) = b$.
(ii) $\exists a \in A$ such that $\forall b \in B,\ f(a) = b$.
(iii) $x, y \in A \wedge (f(x) = f(y)) \Rightarrow x = y$.

One of these statements is the definition for f to be an injective mapping from A to B. Which one is it? One of the statements would be false if B had more then one element. Which one is it? For the remaining statement, please explain what it says and give an example of a function from $A = \{1, 2, 3\}$ to $B = \{1, 2\}$ that has this property.

Solution. (i) means surjective. (ii) is false for B with more than one element. (iii) means injective.

6. (a) Recall that we associate a subset of the natural numbers to a sequence $s = (s_1, s_2, \cdots)$ of 0's and 1's by the assignment

$$Set[s] = \{n \in N | s_n = 1\}.$$

For example

$$Set[(1, 0, 1, 0, 1, 0, \cdots)] = \{1, 3, 5, 7, \cdots\}.$$

Make your best guess about the set associated with the sequence

$$s = (0, 1, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, \cdots).$$

Solution. The prime numbers.

(b) Let $Seq\{0, 1\}$ denote the set of all sequences $s = (s_1, s_2, \cdots)$ of 0's and 1's. Prove that $Seq\{0, 1\}$ is an uncountable set.

Solution. Let

$$s^1, s^2, \cdots$$

be any list of sequences. Each s^i is a sequence in $Seq\{0,1\}$ so that

$$s^i = (s^i_1, s^i_2, s^i_3, \cdots)$$

where s^i_j is equal to 0 or to 1. Now let

$$\bar{\delta} = (1 - s^1_1, 1 - s^2_2, 1 - s^3_3, \cdots).$$

We see that $\bar{\delta}$ is a sequence that differs from every sequence on the list. Hence $Seq\{0,1\}$ is not countable.

7. Let X be any set. Let $P(X)$ denote the set of subsets of X. Let

$$F : X \longrightarrow P(X)$$

be any well-defined mapping from X to its power set $P(X)$. Prove that F is not surjective by exhibiting a subset of X that is not in the image of F.

Solution. Let $C = \{x \in X | x \notin F(x)\}$. Then it follows at once that C is not of the form $F(x)$ for any $x \in X$. For if $C = F(x)$ for some x, then $x \in C$ iff $x \notin F(x)$. But this means $x \in C$ iff $x \notin C$. This is a contradiction, and we conclude that C is not equal to $F(x)$.

Remark. There is a direct relationship between the Cantor diagonal argument and this construction. To see this, consider a mapping

$$F : N \longrightarrow P(N).$$

Then each $F(n)$ for $n = 1, 2, \cdots$ is a subset of the natural numbers, and

$$F(1), F(2), F(3), \cdots$$

is a list of subsets of N. Now let s^n denote the sequence of 0's and 1's that encodes the subset $F(n)$. That is, we have that $s^n_k = 1$ if and only if $k \in F(n)$.

Now apply the Cantor diagonal process to the list of sequences

$$s^1, s^2, \cdots.$$

Thus we define the Cantor anti-diagonal sequence $\bar{\delta}$ by the formula

$$\bar{\delta}_k = 1 - s^k_k.$$

As we know, the anti-diagonal sequence is not equal to any of the sequences s^n. Now, let C denote the set corresponding to $\bar{\delta}$. Note that

$$n \in C \Leftrightarrow \bar{\delta}_n = 1$$

$$\Leftrightarrow 1 - s^n_n = 1$$

$$\Leftrightarrow s^n_n = 0$$

$$\Leftrightarrow n \notin F(n).$$

Thus we have shown that C, the set corresponding to the Cantor anti-diagonal is given by

$$C = \{n \in N | n \notin F(n)\}.$$