Three Extra Credit Problems For Math 215

1. Consider the rectangle diagram shown in the figure below. The corners of the rectangle are $A, B, C, D$ and there is a point $P$ in the rectangle whose distance from $A$ is $a$, from $B$ is $b$, from $C$ is $c$ and from $D$ is $d$. Prove that

\[ a^2 + d^2 = b^2 + c^2. \]

2. Prove that

\[ 2 = (2 + \frac{10}{\sqrt{27}})^{1/3} + (2 - \frac{10}{\sqrt{27}})^{1/3}. \]

In order to do this, note that if $x = a + b$, then

\[ x^3 = a^3 + b^3 + 3ab(a + b). \]

Hence

\[ x^3 = a^3 + b^3 + 3abx. \]

Apply this result to the problem.

3. Let $S$ be a subset of $\{1, 2, 3, \ldots, 2n\}$. Suppose that $|S| = n + 1$. Prove that $S$ contains two numbers such that one number divides the other number. (Hint: Any natural number $m$ can be written in the form $m = (2k - 1)2^j$. That is, it can be written in the form of an odd number times a power of two. For such a number $m$, define $f(m) = k$. Show that this gives a map $f : S \rightarrow \{1, 2, \ldots, n\}$, and make use of this map.)