Fibonacci Form and the Quantized Mark

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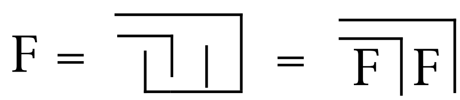
I. Introduction

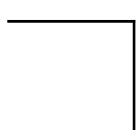
To the extent that this paper is just about the Fibonacci series it is about the Fibonacci Form



whose notation we shall explain in due course. The Fibonacci form is an infinite abstract form that reenters its own indicational space, doubling with a shift that produces a Fibonacci number of divisions of the interior space at every given depth. This is the infinite description of well-known recursions that produce the Fibonacci series.

The paper is organized to provide a wider background for this point of view. We begin by recalling the relationship of the Fibonacci series with the golden ratio and the golden rectangle and we prove a Theorem in Section II showing the uniqueness of the infinite spiraling decomposition of the golden rectangle into distinct squares.

The proof of the Theorem shows how naturally the Fibonacci numbers and the golden ration appear in this classical geometric context. Section III provides background about Laws of Form and the formalism in which the mark



is seen to represent a distinction. With these concepts in place, we are prepared to discuss infinite reentering forms, including the

Fibonacci form in Section IV. Section V continues the discussion of infinite forms and "eigenforms" in the sense of Heinz von Foerster. Finally, Section VI returns to the Fibonacci series via the patterns of self-interaction of the mark. Here the mark is conceived as an "elementary particle" that can interact with itself to either produce itself or to annihilate itself. There are a Fibonacci number of patterns of interaction of a collection of n marks, leading to the unmarked state. This Fibonacci property of the self-interactions of the mark is a link between Laws of Form, topology and quantum information theory. We give the beginning of this relationship in the last secion where we discuss how braiding of quantized marks generates all the unitary transformations one needs for quantum information theory! This is the celebrated Fibonacci model for quantum computation.

II. Geometry of the Golden Ratio

In Figure 1 we have a rectangle of length 377 units and width 233 units. It is paved with squares of sizes

233 x 233, 144 x 144, 89 x 89, 55 x55, 34 x 34, 21 x 21,
13 x13, 8 x 8, 5 x 5, 3 x 3, 2 x 2, 1 x 1, 1 x 1.

This attests to the fact that for the Fibonacci numbers

1,1,2,3,5,8,13,21,34,55,89,144,233,377,

the sum of the squares of the Fibonacci numbers up to a given Fibonacci number is equal to the product of that Fibonacci number with its successor. For example,

$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 = 8 \times 13.$$

Letting
$f(0) = 1$
$f(1) = 1$
$f(2) = 2$
$f(3) = 3$
$f(4) = 5$
and
$f(n+1) = f(n) + f(n-1)$,

we have that a rectangle of size $f(n) \times f(n+1)$ can be paved with squares of different sizes, all except for a repetition

of the size 1×1 at the very end of the spiraling process of cutting off squares that are the width of the given rectangle.
(The fascinating problem of paving a rectangle with squares of all unequal sizes has been studied by Brooks, Smith, Stone and Tutte. See [Tutte1, Tutte2].)

It is well-known that the process of cutting off squares can be continued to infinity if we start with a rectangle that is of the size Phi x1 where Phi is the Golden Ratio Phi = (1 + sqrt{5})/2).

This is no mystery.
Such a process will work when the new rectangle is similar to the original one. That condition is embodied in the equation $W/(L-W) = L/W$, and taking $W=1$, we find $1/(L-1) = L$, whence $L^2 - L - 1 = 0$, whose positive root is the Golden Ratio.

It is also well-known that Phi is the limit of successive ratios of Fibonacci numbers with
$1 < 3/2 < 8/5 < 21/13 < \ldots < \mathrm{Phi} < \ldots < 13/8 < 5/3 < 2.$

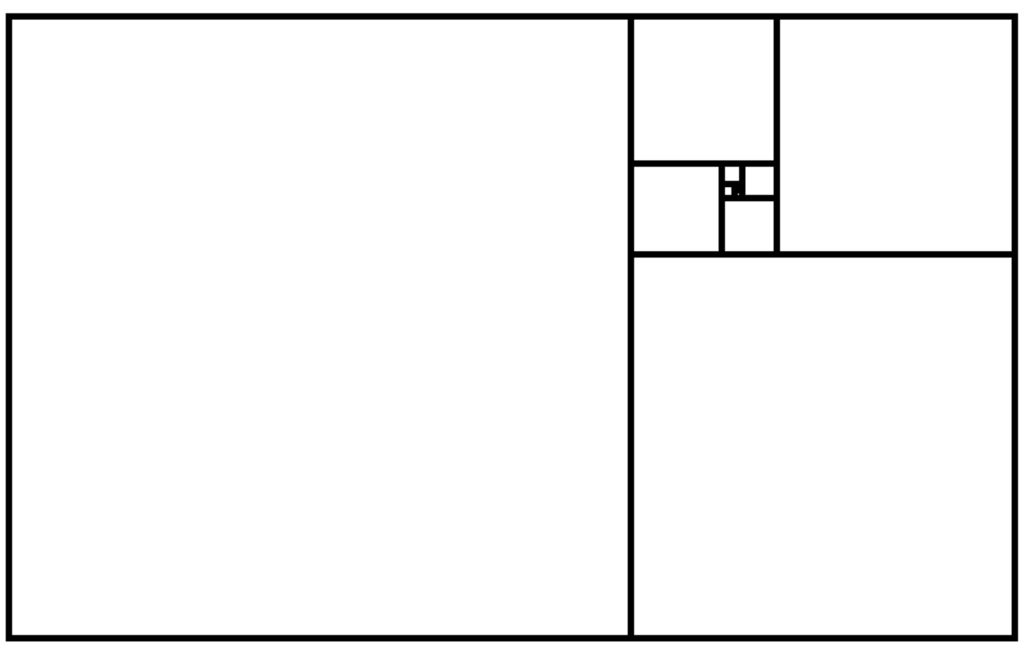


Figure 1 -- The Fibonacci Rectangles

We ask: Is there any other proportion for a rectangle, other than the Golden Proportion, that will allow the process of cutting off successive squares to produce an infinite paving of the original rectangle by squares of different sizes? The answer is: No!

Theorem. The only proportion that allows this pattern of cutting off successive squares to produce an infinite paving of the original rectangle by squares of different sizes is the Golden Ratio.

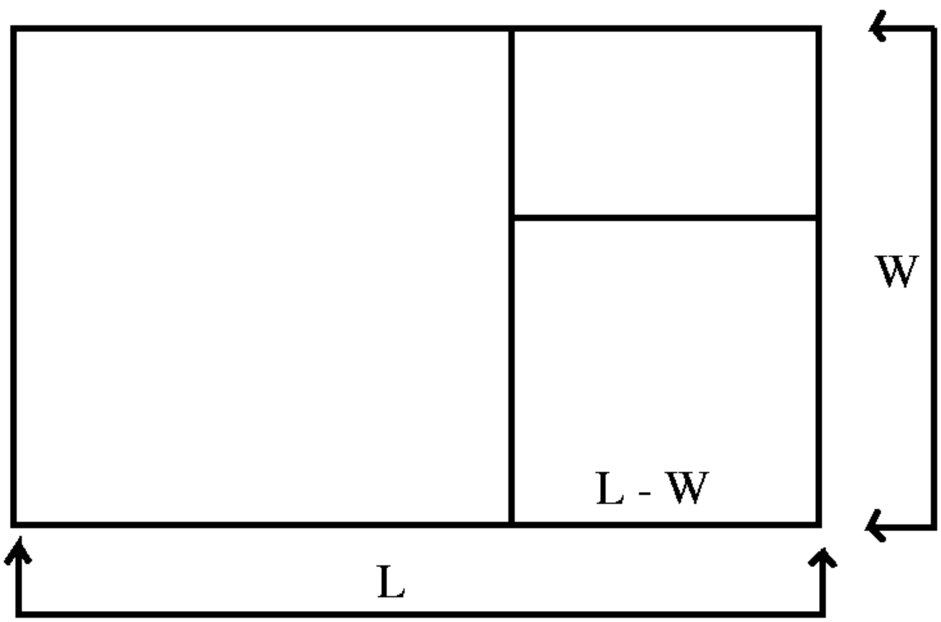


Figure 2 -- Characterizing the Golden Ratio

Proof. View Figure 2.
Suppose the original rectangle has width W and length L.
In order for the process of cutting off the square to produce a single new rectangle whose own square cut-off is smaller than the first square, we need the new rectangle to have width L-W and length W with $L-W < W$.
With this inequality, the square cut off from the new rectangle will be smaller than the first square cut off from the original rectangle.

In order for this pattern of cutting to persist forever, we need an infinite sequence of inequalities, each derived from the previous one. That is, we start with initial length L and width W. We excise a square of size $W \times W$ obtaining a new length $L' = W$ and a new width $W' = L-W$. We require that $W'<L'$ ad infinitum:

$W < L$

$L - W < W$

$W - (L - W) < L - W$
$L - W - (W - (L - W)) < W - (L - W)$
ad infinitum.

Collecting the algebra, we have:
$W < L$
$L - W < W$
$2W - L < L - W$
$2L - 3W < 2W - L$
$5W - 3L < 2L - 3W$
$5L - 8W < 5W - 3L$
$13W - 8L < 5L - 8W$
ad infinitum.
But now the Fibonacci pattern is apparent.
Look at what each of these inequalities says:
$W < L$
$L < 2W$
$3W < 2L$
$3L < 5W$
$8W < 5L$
$8L < 13W$
$21W < 13L$
ad infinitum.
These in turn say:
$L/W > 1$
$L/W < 2$
$L/W > 3/2$
$L/W < 5/3$
$L/W > 8/5$
$L/W < 13/8$
$L/W > 21/13$
ad infinitum.

Thus we see from this pattern that L/W is sandwiched between ratios of Fibonacci numbers just as we know the Golden Ratio is sandwiched:
$1 < 3/2 < 8/5 < 21/13 < \ldots < L/W < \ldots < 13/8 < 5/3 < 2$,
and this implies that $L/W = \Phi = (1 + \sqrt{5})/2$.
The rectangle is Golden. QED

Here is another take on the Theorem. As above, let
$L' = W$ and $W' = L - W$ with the assumption that
$W < L$ and $L - W < W$. Then

$L/W = (W + (L-W))/W = 1 + (L-W)/W = 1 + 1/(W/(L-W)).$

Thus

$L/W = 1 + 1/(L/W)'$

where $(L/W)' = L'/W'$.
We see that if we define $P = P_0 = L/W$ and $P_{n+1} = ((L/W)_n)'$ then

$P = 1 + 1/P_1 = 1 + 1/(1 + 1/P_2)$
$= 1 + 1/(1 + 1/(1 + 1/P_3)) = \ldots$

If the P_n have a limiting value, then

$L/W = P = 1 + 1/(1 + 1/(1 + 1/\ldots)).$

In fact we have proved that $(L/W)' = L/W$. In this way we recapture the well-known formula for the golden ratio as a continued fraction.

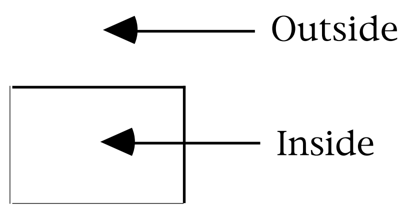
$Phi = 1 + 1/(1 + 1/(1 + 1/\ldots)).$

Note again that we did not start with the assumption that $L/W = L'/W'$. We proved that the Golden Proportion follows from the assumption that one can continue the dissection into squares ad infinitum.

III. Laws of Form

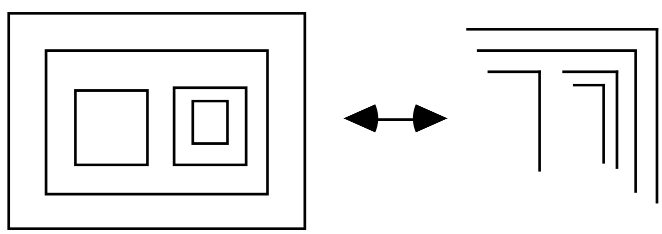
"Laws of Form" by George Spencer-Brown [LOF] is a lucid book with a topological notation based on one symbol, the mark:

This single symbol represents a distinction between its inside and its outside:

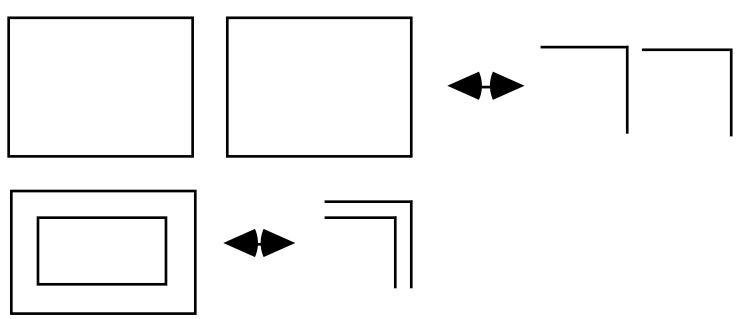


As is evident from the figure above, the mark is regarded as a shorthand for a rectangle drawn in the plane and dividing the plane into the regions inside and outside the rectangle.

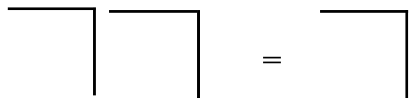
In this notation the idea of a distinction is instantiated in the distinction that the mark makes in the plane. Patterns of non-intersecting marks (that is non-intersecting rectangles) are called *expressions*. For example,

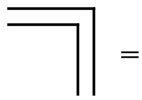


In this example, I have illustrated both the rectangle and the marked version of the expression. In an expression you can say definitively of any two marks whether one is or is not inside the other. The relationship between two marks is either that one is inside the other, or that neither is inside the other. These two conditions correspond to the two elementary expressions shown below.



The mathematics in Laws of Form begins with two laws of transformation about these two basic expressions. Symbolically, these laws are:





In the first of these equations (the law of calling) two adjacent marks condense to a single mark, or a single mark expands to form two adjacent marks. In the second equation (the law of crossing) two marks, one inside the other, disappear to form the unmarked state indicated by nothing at all. Alternatively, the unmarked state can give birth to two nested marks. A calculus is born of these equations, and the mathematics can begin.

Spencer-Brown begins his book, before introducing this notation, with a chapter on the concept of a distinction.

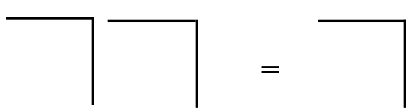
"*We take as given the idea of a distinction and the idea of an indication, and that it is not possible to make an indication without drawing a distinction. We take therefore the form of distinction for the form.*"

From here he elucidates two laws:

1. The value of a call made again is the value of the call.
2. The value of a crossing made again is not the value of the crossing.

The two symbolic equations above correspond to these laws. The way in which they correspond is worth discussion.

First look at the law of calling. It says that the value of a repeated name is the value of the name. In the equation



one can view either mark as the name of the state indicated by the outside of the other mark.

In the other equation

$$\overline{\overline{}|}| =$$

the state indicated by the outside of a mark is the state obtained by crossing from the state indicated on the inside of the mark. Since the marked state is indicated on the inside, the outside must indicate the unmarked state. The Law of Crossing indicates how opposite forms can fit into one another and vanish into the Void, or how the Void can produce opposite and distinct forms that fit one another, hand in glove.

The same interpretation yields the equation

$$\overline{}| = \overline{}|$$

where the left-hand side is seen as an instruction to cross from the unmarked state, and the right hand side is seen as an indicator of the marked state. The mark has a double carry of meaning. It can be seen as an operator, transforming the state on its inside to a different state on its outside, and it can be seen as the name of the marked state. That combination of meanings is compatible in this interpretation.

From indications and their calculus, one moves to algebra where it is understood that a variable is the conjectured presence or absence of a mark. Thus

$$\overline{\overline{A}|}|$$

stands for the two possibilities

$$\overline{\overline{\overline{}|}|}| = \overline{}| \quad , A = \overline{}|$$

$$\overline{\overline{}|}| = \quad\quad , A =$$

In all cases of A we have

$$\overline{\overline{A}|}| = A .$$

Thus begins algebra with respect to this non-numerical arithmetic of forms. The primary algebra that emerges is a subtle precursor to Boolean algebra.

Other examples of algebraic rules are the following:

$$aa = a$$

$$\overline{\quad}|\, a = \overline{\quad}|$$

$$\overline{\overline{a}|\, a}| =$$

$$\overline{ab}|\, b = \overline{\ a\ }|\, b$$

Each of these rules is easy to understand from the point of view of the arithmetic of the mark. Just ask what you will get if you substitute values of **a** and **b** into the equation. For example, in the last equation, if **a** is marked and **b** is unmarked, then the equation becomes

$$\overline{\overline{\quad}|\,\overline{\quad}|}|\,\overline{\quad}| = \overline{\overline{\quad}|\qquad}|\,\overline{\quad}|$$

which is certainly true, by the law of calling.

With algebra one can solve equations, and Spencer-Brown pointed out that one should consider equations of higher degree in the primary algebra just as one does in elementary algebra.

Such equations can involve self-reference. Lets look at ordinary algebra for a moment.

$$x^2 = ax + b$$

is a quadratic equation with a well-known solution, and it is also well-known that the solution is sometimes imaginary in the sense that it utilizes complex numbers of the form $R + Si$ where $i^2 = -1$. One can re-write the equation as

$$x = a + b/x.$$

In this form it is indeed self-referential, with x re-entering the expression on the right. We could "solve" it by an infinite reentry or infinite continued fraction:

$$x = a + b/(a + b/(a + b/(a + b/(a + b/(a + ...))))).$$

In this infinite formalism it is literally the case that $x = a + b/x$ and we can write

$$a + b/(a+b/(a+...))) = [a+ b/ \;]$$

to indicate how this form reenters its own indicational space.
This formal solution to the quadratic equation converges to a real solution when the quadratic equation has real roots. For example, if $a=1=b$, then

$$1+1/(1+1/(1+...))) = [1+ 1/ \;]$$

converges to the positive solution of $x^2 = x + 1$, which is the golden
ratio, φ = (1 + _5)/2.

On the other hand, the quadratic equation may have imaginary roots. (This happens when $a^2 + 4b$ is less than zero.) Under these circumstances, the formal solution does not represent a real number.
For example, if i denotes the square root of minus one, then we could write

$$i = -1/(-1/(-1/...)) = [-1/ \;]$$

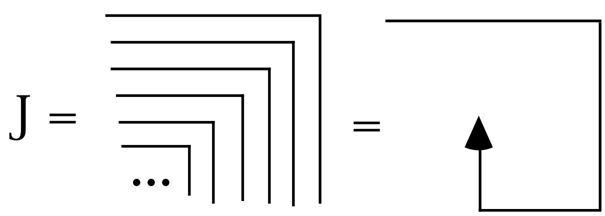
to denote a formal number with the property that

$$i = -1/i .$$

Spencer-Brown makes the point that one can follow the analogy of introducing imaginary numbers in ordinary algebra to introduce *imaginary boolean values* in the arithmetic of logic.
An apparently paradoxical equation such as

$$J = \overline{J\,}\big|$$

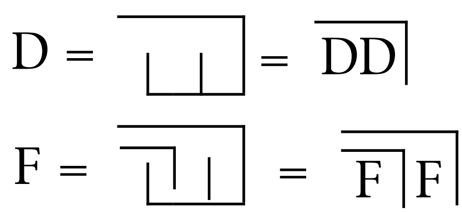
can be regarded as an analog of the quadratic $x = -1/x$, and its solutions will be values that go beyond marked and unmarked, beyond true and false.



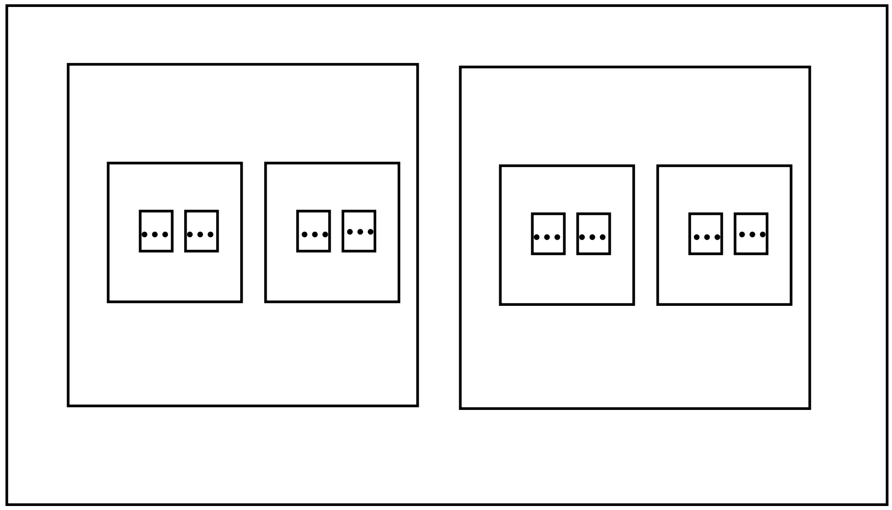
Sometimes one represents J as an infinite form that reenters its own indicational space.

IV. Infinite Recursive Forms

Constructions involving the mark, suggest considering all possible expressions, including infinite expressions, with no arithmetic initials other than commutativity of juxtapositions.
We shall call such expressions *forms*. Here we shall discuss some of the phenomenology of infinite forms that are described by reentry.
This simplest example of such a form is the reentering mark J as discussed above. Here are the next two simplest examples.



I call D the *doubling form*, and F the *Fibonacci form*.
A look at the recursive approximations to D shows immediately why we have called it the doubling form (approximations are done in box form):

D = 

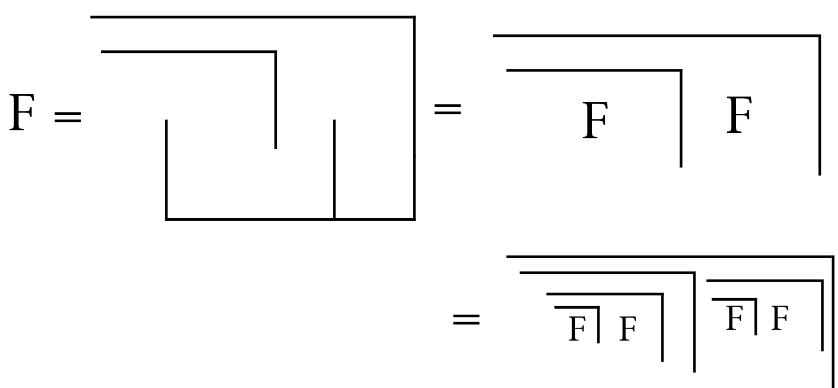
We see from looking at the approximations, that the number of divisions of D doubles at each successive depth beyond depth zero. Letting D_n denote the number of divisions of D at depth n, we see that $D_n = 2^{n-1}$.

Given any forms G and H, we define G_n to be the number of divisions of G at depth n. We have the basic formulas:

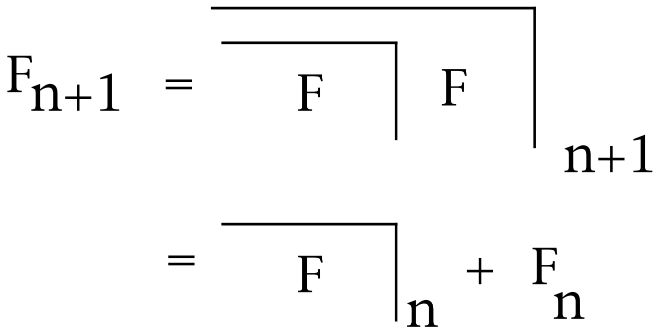
$$\overline{G|}_{n+1} = G_n$$

$$(GH)_n = G_n + H_n$$

In the case of the Fibonacci form, we have



Thus

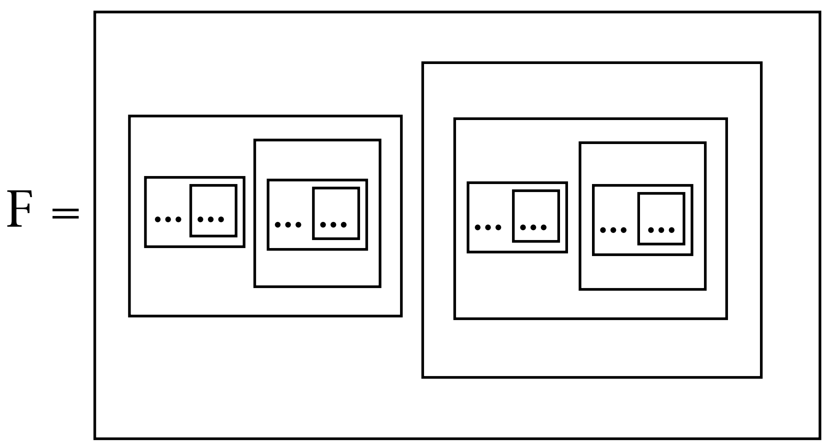


$$F_{n+1} = F_{n-1} + F_n$$

For the Fibonacci form, $F_{n+1} = F_n + F_{n-1}$ with $F_0=F_1=1$.
The depth counts in this form are the Fibonacci numbers

1,1,2,3,5,8,13,21,34,55,89,144,...

with each number the sum of the preceding two numbers.



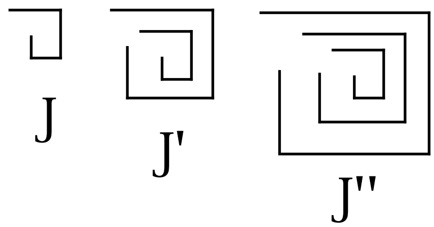
The Fibonacci Form

It is natural to define the *growth rate* $\mu(G)$ of a form G to be limit of the ratios of successive depth counts as the depth goes to infinity.

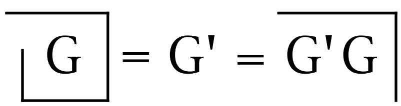
$$\mu(G) = \lim_{n \text{-->Infinity}} G_{n+1}/G_n.$$

Then we have $\mu(D) = 2$, and $\mu(F) = (1 + \_5)/2$, the golden ratio.

Finally, here is a natural hierarchy of recursive forms, obtained each from the previous by enfolding one more reentry.



Given any form G, we define G' by the formula shown below, so that



$$G'_{n+1} = G'_n + G_{n-1}$$

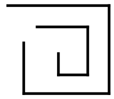
This implies that

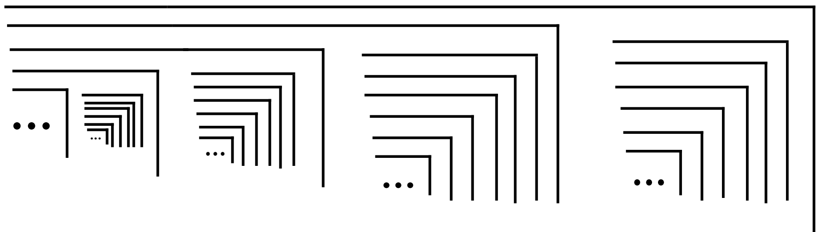
$G'_{n+1} - G'_n = G_{n-1}$.

Thus the discrete difference of the depth series for G' is (with a shift) the depth series for G. The series J, J', J'', J''', ... is particularly interesting because:

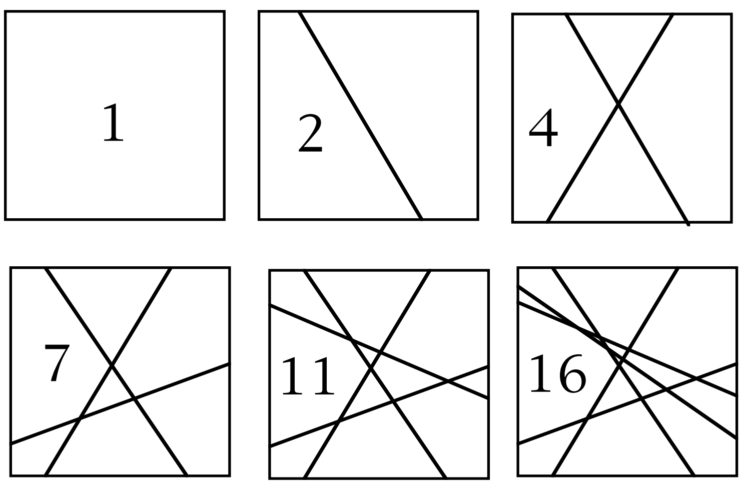
The depth sequence $(J^{(n)})_k$ is equal to the maximal number of divisions of n-dimensional Euclidean space by $k-1$ hyperspaces of dimension $n-1$.

We will not prove this result here, but note that J takes the role of a point (dimension zero) with $J_k = 1$ for all k,
while J' satisfies $J'_{k+1} = J'_k + 1$ $(k > 0)$, so that
$J'_k = k-1$ for $k>1$.
This is the formula for the number of divisions of a line by $k-1$ points.

J' = 

= 

To think about the divisions of hyperspace, think about how a collection of lines in general position in the plane intersect one another. If a new line is placed, it will cut a number of regions into two regions. The number of new regions is equal to the number of divisions made in the new line itself. This is a verbal description of the basic recursion given above.



$1 + 1 = 2$

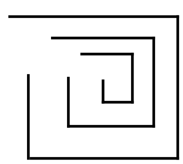
$2 + 2 = 4$

$4 + 3 = 7$

$7 + 4 = 11$

$11 + 5 = 16$

...

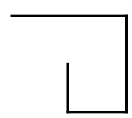


The very simplest recursive forms yield a rich complexity of behaviours that lead directly into the mathematics of imaginary numbers and oscillations, patterns of growth, dimensions and geometry..

There is an eternity and a spirit at the center of each complex form. That eternity may be an idealization, a "fill-in", but it is nevertheless real. In the end it is that eternity, that eigenform unfolding the present moment that is all that we have. We know each other through our idealizations of the other. We know ourselves through our idealization of ourselves. We become what we were from the beginning, a Sign of Itself [P, MP] .

V. Eigenforms

Consider the reentering mark.



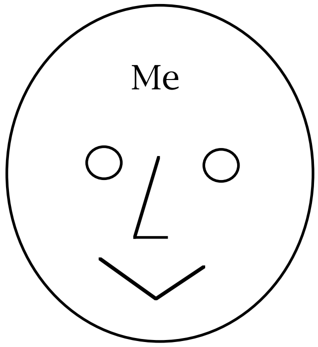
This is an archetypal example of an eigenform in the sense of Heinz von Foerster [VF]. An eigenform is a solution to an equation, a solution that occurs at the level of form, not at the level of number. You live in a world of eigenforms. You thought that those forms you see are actually "out there"? Out where? The very space, the context that you regard as your external world is an eigenform. It is your organism's solution to the problem of distinguishing itself in a world of actions.

The shifting boundary of Myself/MyWorld is the dynamics of the form that "you" are. The reentering mark is the solution to the equation

$$J = \overline{\;\;J\;\;}|$$

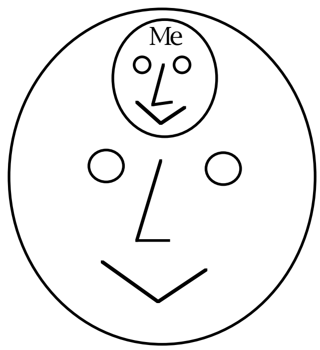
where the right-angle bracket distinguishes a space in the plane. This is not a numerical equation. One does not even need to know any particularities about the behaviour of the mark to have this equation. It is akin to solving

Me =



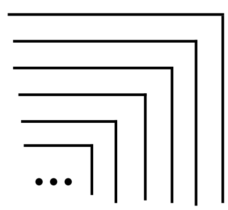
by attempting to create a space where "I" can be both myself and inside myself, as is true of our locus psychological. And this can be solved by an infinite regress of Me's inside of Me's.

Me =



Just so we may solve the equation for J by an infinite nest of boxes

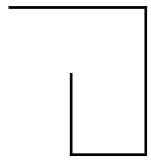
J =



Note that in this form of the solution, layered like an onion, the whole infinite form reenters its own indicational space. It is indeed a solution to the equation

$$J = \overline{\;J\;}|$$

The solution in the form

J = 

is meant to indicate how the form reenters its own indicational space. This reentry notation is due to G. Spencer-Brown. Although he did not write down the reentering mark itself in his book "Laws of Form", it is implicit in the discussion in Chapter 11 of that book.

It is not obvious that we should take an infinite regress as a model for the way we are in the world. Everyone has experienced being between two reflecting mirrors and the veritable infinite regress that arises at once in that situation. Physical processes can happen more rapidly than the speed of our discursive thought, and thereby provide ground for an excursion to infinity.

Here is one more example. This is the eigenform of the Koch fractal [SRF,EF]. In this case one can write the eigenform equation

$$K = K \{ K \quad K \} K.$$

The curly brackets in the center of this
equation refer to the fact that the two middle copies within the fractal are inclined with respect to one another and with respect to the two outer copies. In the figure below we show the geometric configuration of the reentry.

The Koch fractal reenters its own indicational space **four** times (that is, it is made up of four copies of itself, each **one-third** the size of the original. We say that the Koch fractal has *replication rate* **four** and write $R(K)=4$. We say it has *length ratio* **three** and write $F(K)=3$.

In describing the fractal recursively, one starts with a segment of a given length L. This is replaced by a $R(K)$ segments each of length $L' = L/F(K)$. In the equation above we see that $R(K)=4$ is the number of reentries, and $F(K)$ is the number of groupings in the reentry form.

It is worth mentioning that the fractal dimension D of a fractal such as the Koch curve is given by the formula

$$D = \ln(R)/\ln(F)$$

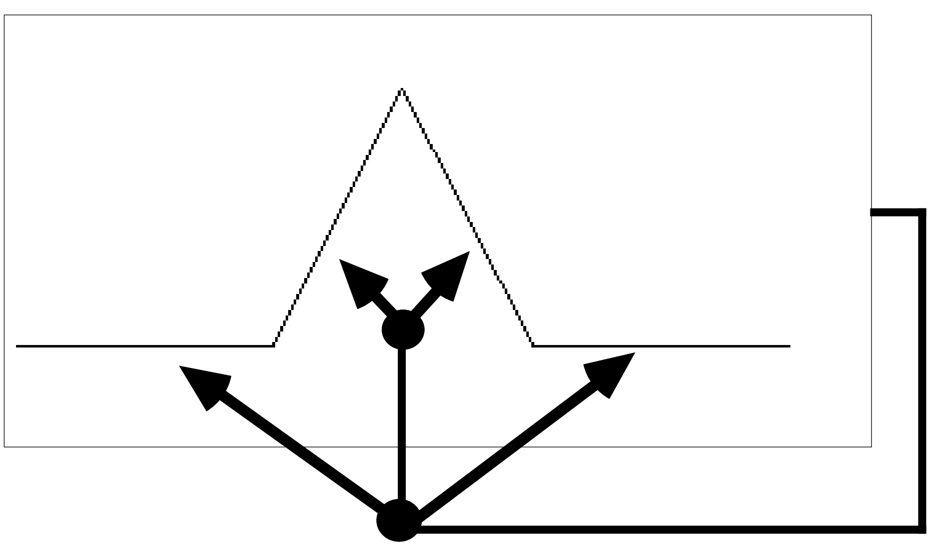
where R is the replication rate of the curve , F is the length ratio and $\ln(x)$ is the natural logarithm of x.

In the case of the Koch curve one has $D = \ln(4)/\ln(3)$. The fractal dimension measures the fuzziness of the limit curve. For curves in the plane, this can vary between 1 and 2, with curves of dimension two having space-filling properties.

It is worth noting that we have, the case of an abstract, grouped reentry form such as K = K { K K } K, a corresponding abstract notion of fractal dimension, as described above

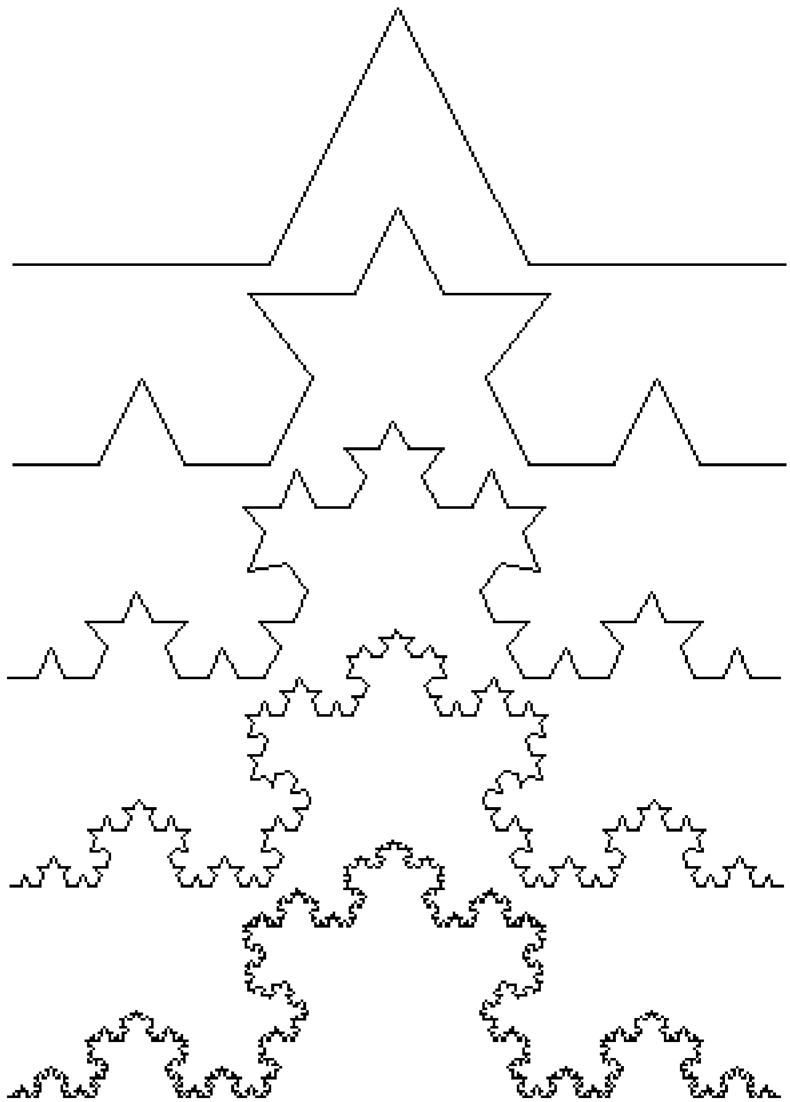
D(K) = ln(Number of Reentries)/ln(Number of Groupings).

As this example shows, this abstract notion of dimension interfaces with the actual geometric fractal dimension in the case of appropriate geometric realizations of the form. There is more to investigate in this interface between reentry form and fractal form.



K = K { K K } K

In the geometric recursion, each line segment at a given stage is replaced by four line segments of one third its length, arranged according to the pattern of reentry as shown in the figure above. The recursion corresponding to the Koch eigenform is illustrated in the next figure. Here we see the sequence of approximations leading to the infinite self-reflecting eigenform that is known as the Koch snowflake fractal.



Five stages of recursion are shown. To the eye, the last stage vividly illustrates how the ideal fractal form contains four copies of itself, each one-third the size of the whole. The abstract schema

K = K { K K } K

for this fractal can itself be iterated to produce a "skeleton" of the geometric recursion:

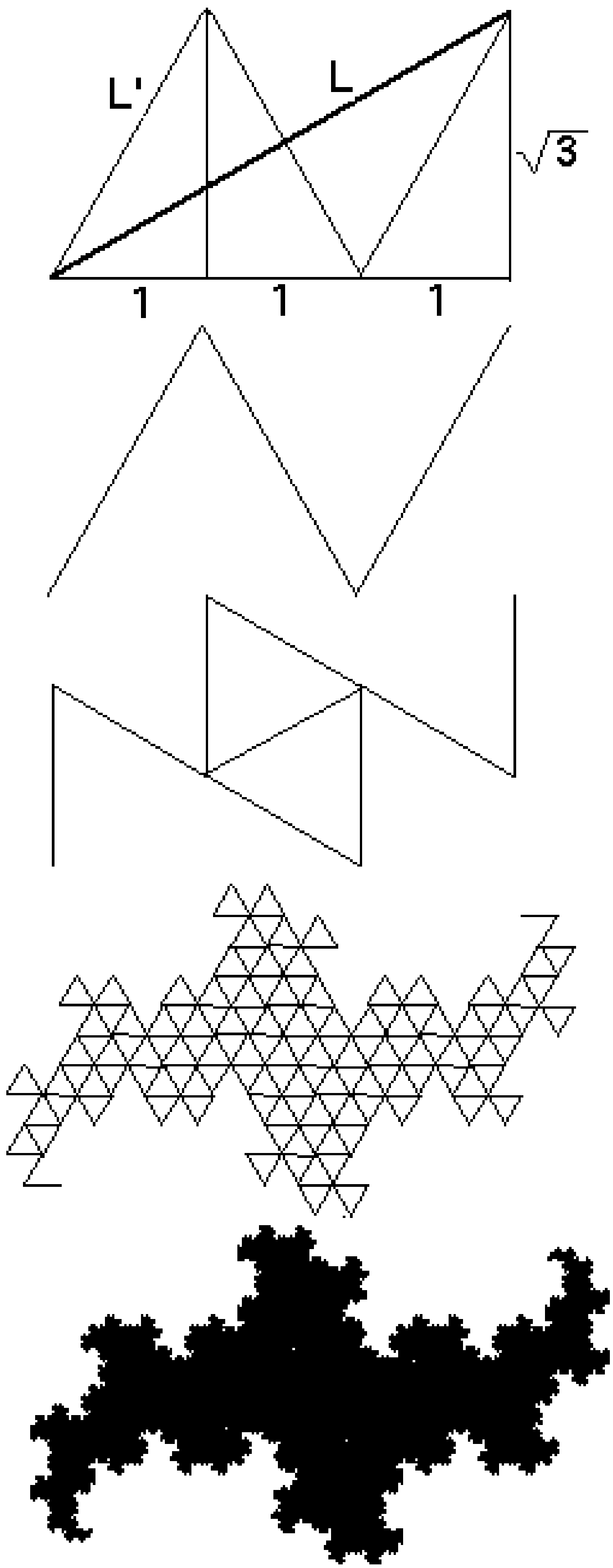
K = K { K K } K

= K { K K } K { K { K K } K K { K K } K } K { K K } K

= ...

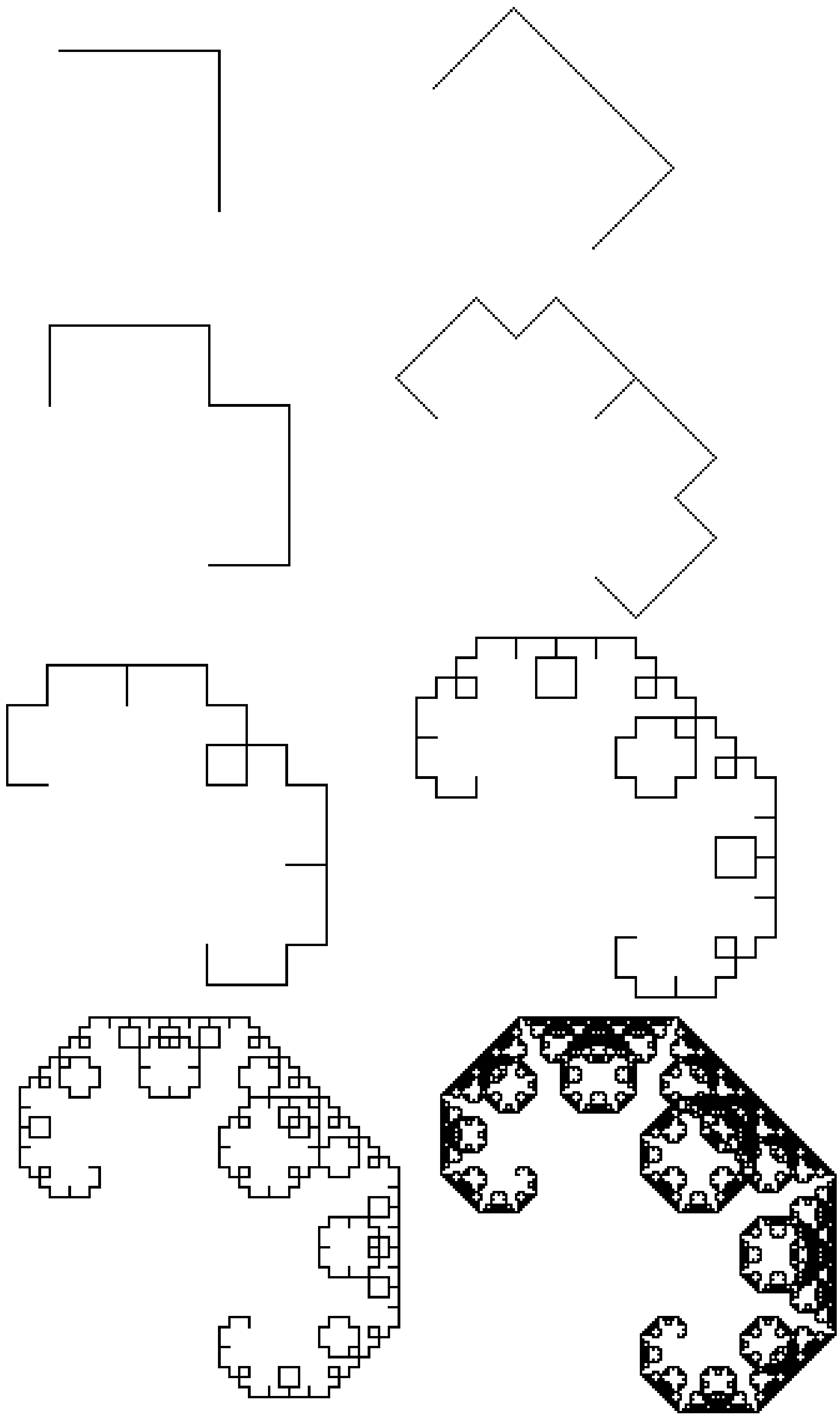
We have only performed one line of this skeletal recursion. There are sixteen K's in this second expression just as there are sixteen line segments in the second stage of the geometric recursion. Comparison
with this abstract symbolic recursion shows how geometry aids the intuition.

Geometry is much deeper and more surprising than the skeletal forms. The next example illustrates this very well. Here we have the initial length L being replaced by a length three copies of L' with L/L' equal to the square root of 3. (To see that L/L' is the square root of three, refer to the illustration below and note that L'= _(1 + 3) = 2, while L = _(9 + 3) = 2_3.) Thus this fractal curve has dimension D = ln(3)/ln(_3) = 2. In fact, it is strikingly clear from the illustration that the curve is space-filling. It tiles its interior space with rectangles and has another fractal curve as the boundary limit.


L'
L
$\sqrt{3}$
1
1
1

The interaction of eigenforms with the geometry of physical, mental, symbolic and spiritual landscapes is an entire subject that is in need of deep exploration. Compare with [EF].

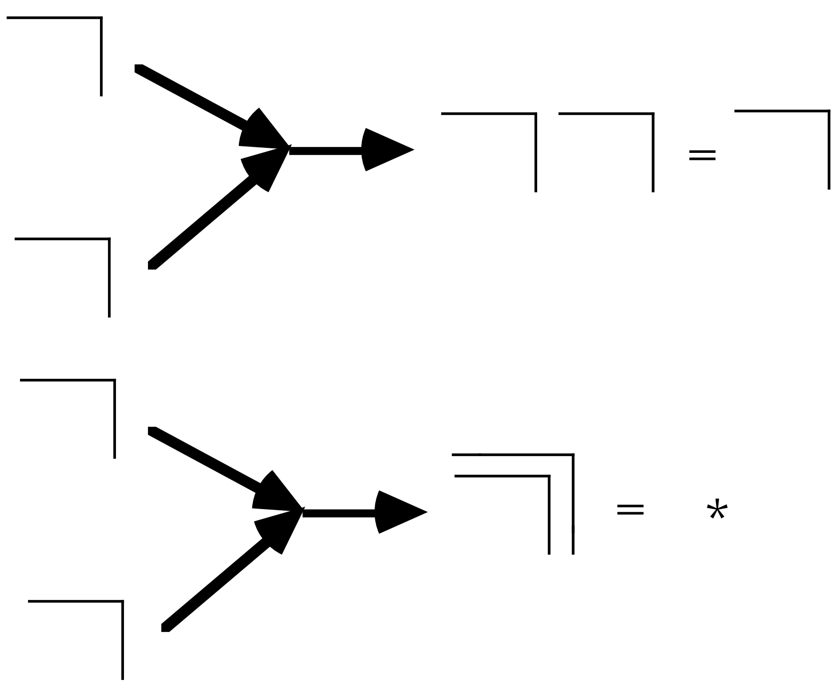
As a last fractal example for this section, here is a beautiful specimen SB generated by the Spencer-Brown mark. That is, the generator for this fractal is a ninety degree bend. Each segment is replaced by two segments at ninety degrees to one another, and the ratio of old segment to new segment is _2. Thus we have D(SB) = ln(2)/ln(_2) = 2, another space-filler. Notice how in the end, we have an infinite form that is a superposition of two smaller copies of itself at ninety degrees to one another.



It is usually thought that the miracle of recognition of an object arises in some simple way from the assumed existence of the object and the action of our perceiving systems. What is to be appreciated is that this is a fine tuning to the point where the action of the perceiver, and the perception of the object are indistinguishable. Such tuning requires an intermixing of the perceiver and the perceived that goes beyond description. Yet in the mathematical levels, such as number or fractal pattern, part of the process is slowed down to the point where we can begin to apprehend it. There is a stability in the comparison, in the one-to-one correspondence that is a process happening at once in the present time. The closed loop of perception occurs in the eternity of present individual time. Each such process depends upon linked and ongoing eigenbehaviors and yet is seen as simple by the perceiving mind.

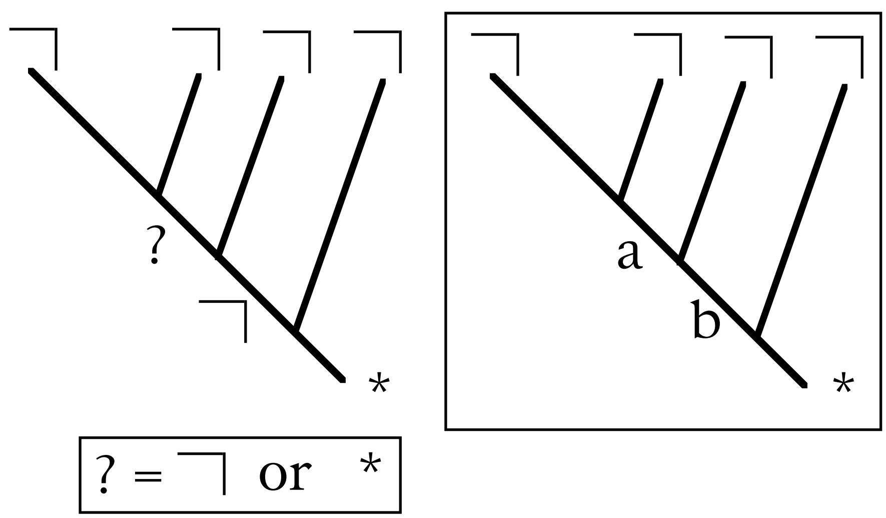
VI. Fibonacci Particles

Think of the Spencer-Brown mark as an "elementary particle" that has two modes of interaction. Two marks can interact to produce either one mark or nothing.



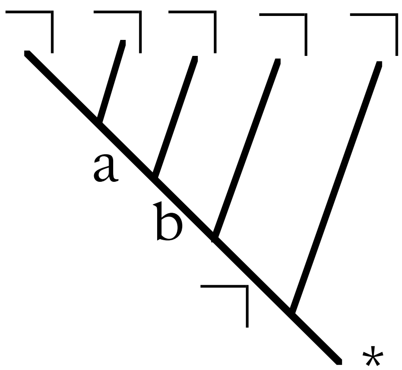
Here we have indicated the interactions, with * denoting the unmarked state. An entity whose only interaction is to produce itself or annihilate itself is surely the simplest non-trivial elementary particle! The mark embodies this pattern by the choice, for two marks that they interact via calling or crossing. The choice at the level of distinctions is the question whether one distinction is inside or outside of the other.

Here is a way to interface Laws of Form and quantum theory, by studying the structure of this single-particle theory. The purpose of including this structure in the present paper is its relationship with the Fibonacci numbers. We look at the possible successive interations of this particle. Consider the diagram below.



In this diagram we have illustrated four initial particles that are to interact in the pattern shown above. That is, the left two particles interact to produce the question mark (which can be either marked or unmarked). Then the question mark interacts with a mark to produce a mark and the mark interacts with the fourth mark across the top to produce an unmarked state, shown at the bottom of the diagram. If we want an unmarked state to appear at the bottom of the diagram then the last interaction must be between two marks, since an unmarked state interacting with a marked state can produce only a marked state. The question mark can be either marked or unmarked to accomplish the overall pattern. In the box next to this process diagram we have indicated the form that was filled in. In the form we have unknown states **a** and **b** that can be either marked or unmarked. We see that the solutions to the possibilities for **a** and **b** are $(a,b) = (U, M)$ and $(a,b) = (M,M)$ where U stands for the unmarked state, and M stands for the marked state. With two solutions, we say that this space of processes is two-dimensional.

Now consider the processes that will solve the analogous problem with five initial marks.

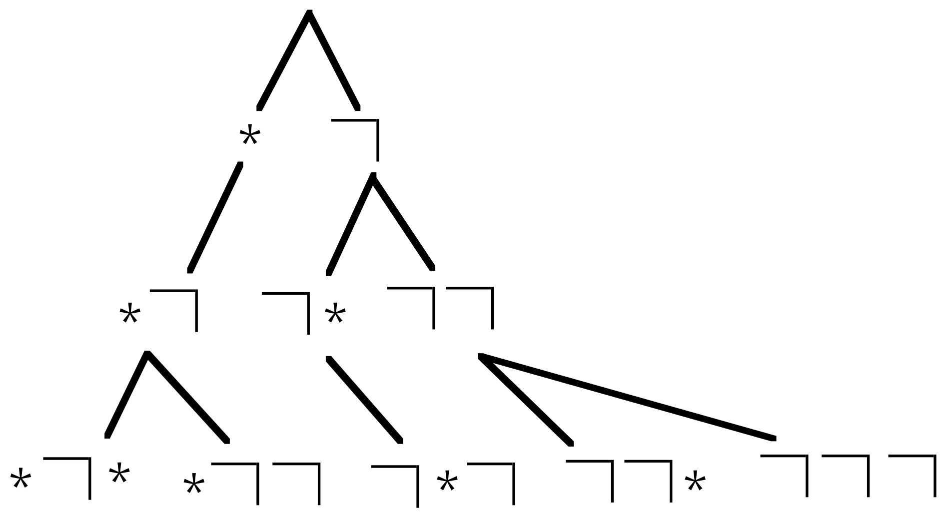


(a,b) = (⌉,⌉)
(⌉,*)
(* ,⌉)

$\dim(V(5)) = 3 = f(3)$

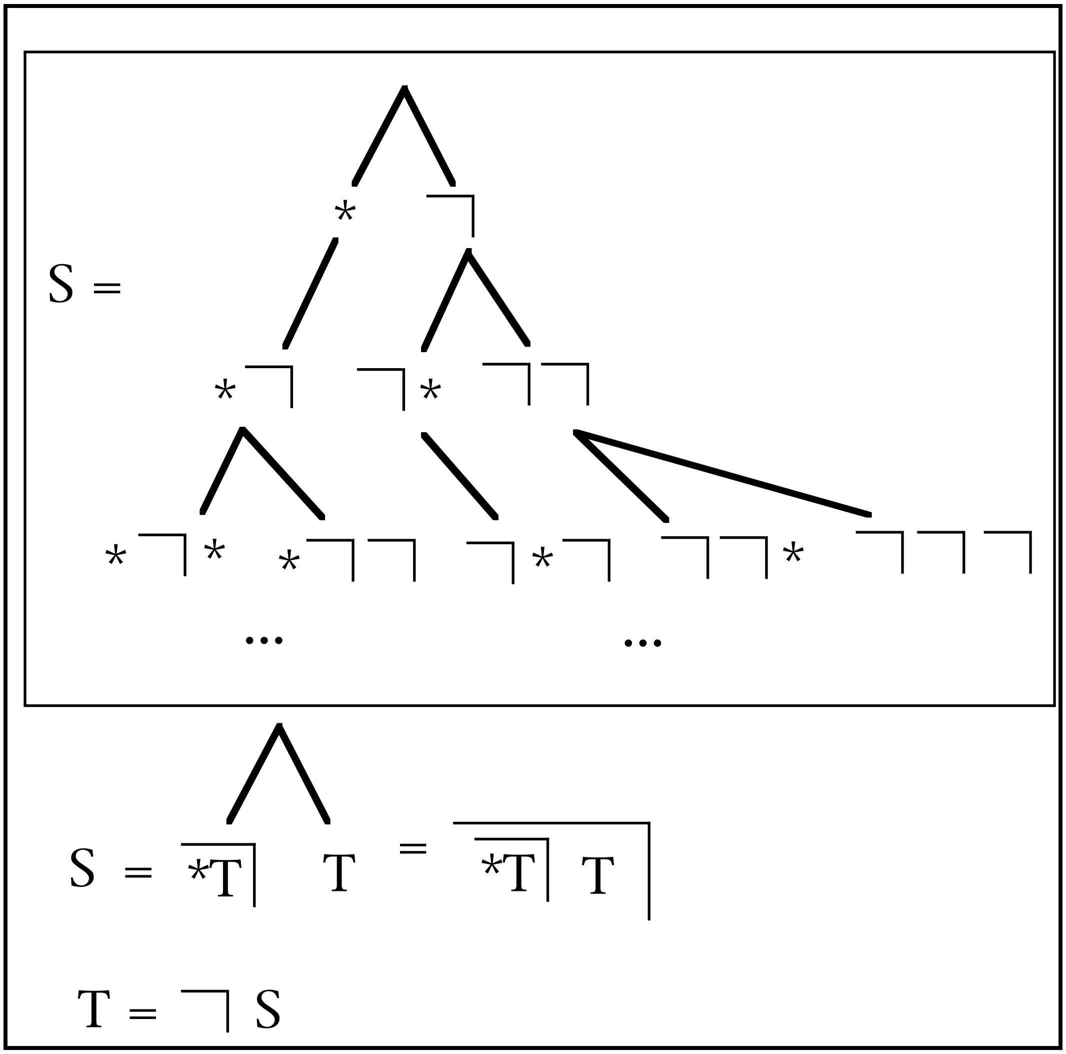
Now we get sequences of the form (a,b,M) and we see that the solutions (a,b) are: (M,M), (M,U), (U,M). One thing that will not do is $(a,b) = (U,U)$. We cannot have two consecutive unmarked states in this game, since any give unmarked state will interact with one of the initial marks to produce a marked state. Thus we see that for a general process space with $n + 2$ initial marks, and ending with the unmarked state, we will solve it with sequences $(a_1,a_2,\ldots,a_n, M)$ where in the sequence $a = (a_1,a_2,\ldots,a_n)$ we can have arbitrary choices of marked and unmarked states with the stipulation that *no two consecutive terms are both unmarked.*

In this way we are led to consider sequences of marked and unmarked states such that no two consecutive elements of the sequence are unmarked. It is easy to see that the number of such sequences of length n is the $n+1$-th Fibonacci number $f(n+1)$ where
$f(0) = 1$, $f(1) = 1$, $f(2) = 2$ and so on.



The upshot is that the dimension of the space V(n+2) of interactions of n+2 elementary marked particles to produce a single unmarked particle is the Fibonacci number f(n).

Note that the form of the infinite tree indicated above is a division into two infinite trees (the one below * and the one below the mark) with the left tree obtained from the right tree by shifting it down one level and placing a star on the left of every sequence in the right tree. The right tree we shall call T, and the whole tree S. Then it is the case that the right tree is obtained from the whole tree by putting a mark to the left of each sequence on the whole. Thus we have the situation diagrammed below. In writing equations we have used a mark *over* a symbol to denote the down-shifting of the corresponding tree structure.

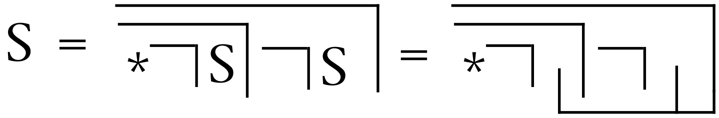


We can combine these equations to obtain

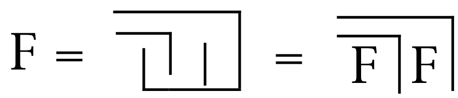
$$S = \overline{\overline{*T}|\; T}\,|$$

$$T = \neg\, S$$

Hence



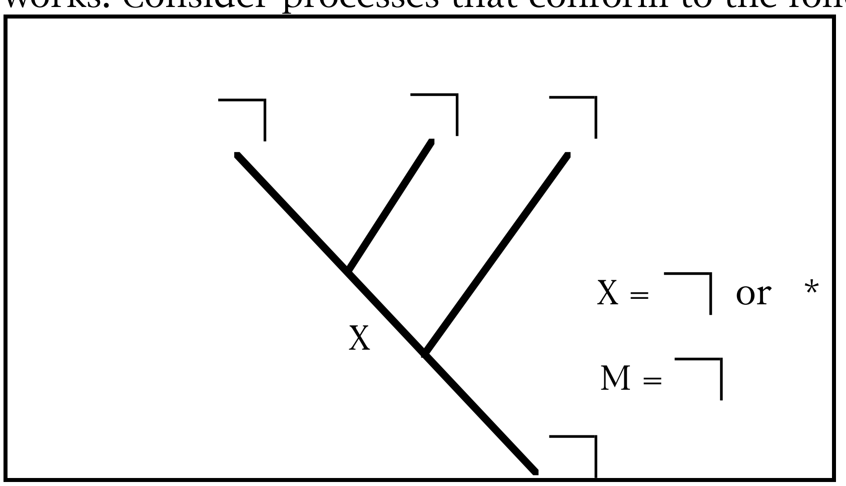
and with this reentry form for the Fibonacci sequences, we return again to the pattern of the Fibonacci Form.



This natural appearance of the Fibonacci Numbers and the Fibonacci Form in the self-interactions of the mark has far-reaching consequences. It turns out that there is an intimate relationship of the properties of this model with certain unitary representations of the braid group, and that these representations can be used to generate a robust set of unitary matrices. These matrices can, in turn be configured as a topological quantum field theory, and this field theory can model the operations of a quantum computer. See [KF, KP, Preskill] for more details of this story of the Fibonacci Anyons. The Fibonacci Series gets around.

VII. The Quantized Mark

We will give a hint here about how this unitary representation works. Consider processes that conform to the following tree.



Threre are only two possibilities here. The two initial marks on the left either interact to procduce a mark, or they interact to produce a star. This production, X, then is constrained to interact with the second mark to produce a mark. Thus the space of possibilities is two dimensional. We can write M for the mark and write
|X> for the two basis states for this space. Then we have that

This space of processes has basis { |*>, |M> }. It is a two dimesional space and we can regard it as a single-qubit space, the beginning of quantum computation.

Call the space spanned by these proceeses H.
A qubit is a vector in this space with the form

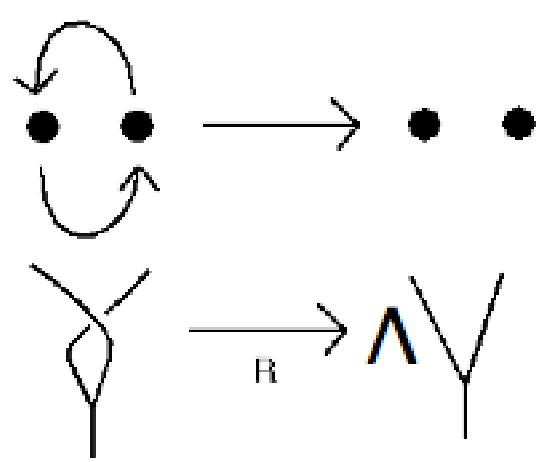
$$|v\rangle = a\,|*\rangle + b|M\rangle$$

where a and and b are complex numbers such that $|a|^2 + |b|^2 = 1$.
In the quantum model, an observaton of |v> results in either |*> or |M> with the probability of observing |*> equal to $|a|^2$ and the probability of observing |M> equal to $|b|^2$. This sort of probability is preserved by unitary transformations of the space H. A uniitary transformation is represented by a 2 x 2 matrix U such that

$$U^* = U^{-1}$$

where U* denotes the conjugate transpose of U (transpose the matrix and take the complex conjugates of its entries). In the quantum model, all (unobserved) physical processes are represented by unitary transformations of the state space H.
Of course one often needs higher dimensional spaces and these can be obtained by taking tensor products of H with copies of itself.
The same principles about measurement and unitarity apply to the higher dimensional spaces. This is quantum mechanics in a nutshell!

Now we have structured our space H so that it corresponds to interactions of three marks in the associated order (MM)M as in the tree of the last figure. The tree itself suggests that maybe we could consider braiding of the particles.

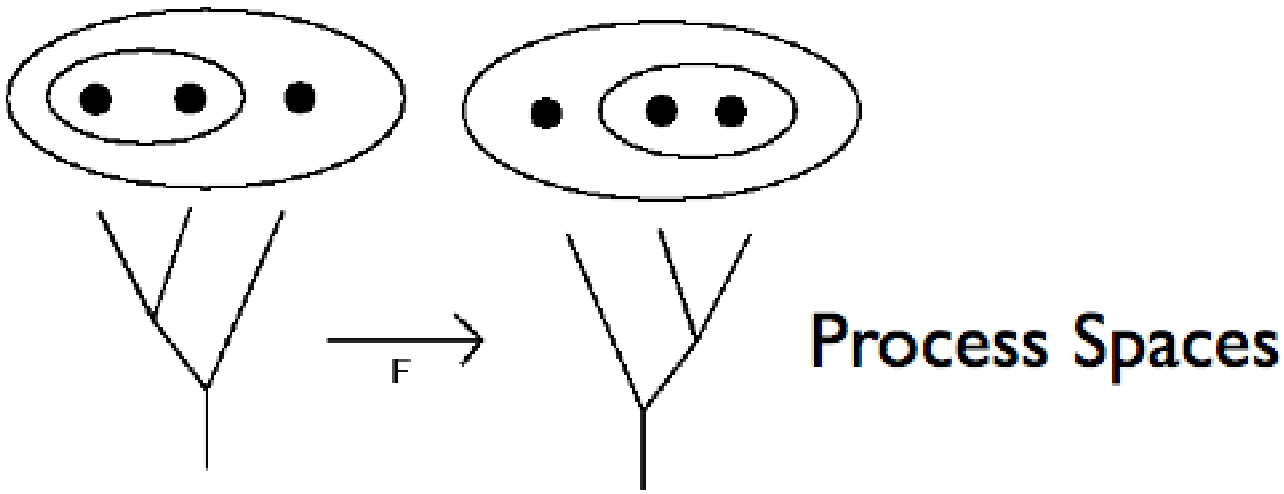


Because we envisage these particles as interacting in a plane space, it is possible for the interchange of two particles to give rise to a phase change in the quantum wave function for them. We take the wave function that represents the process for the particle interaction. Thus a different phase could occur if M interacts with itself to produce * or to produce M. We can write

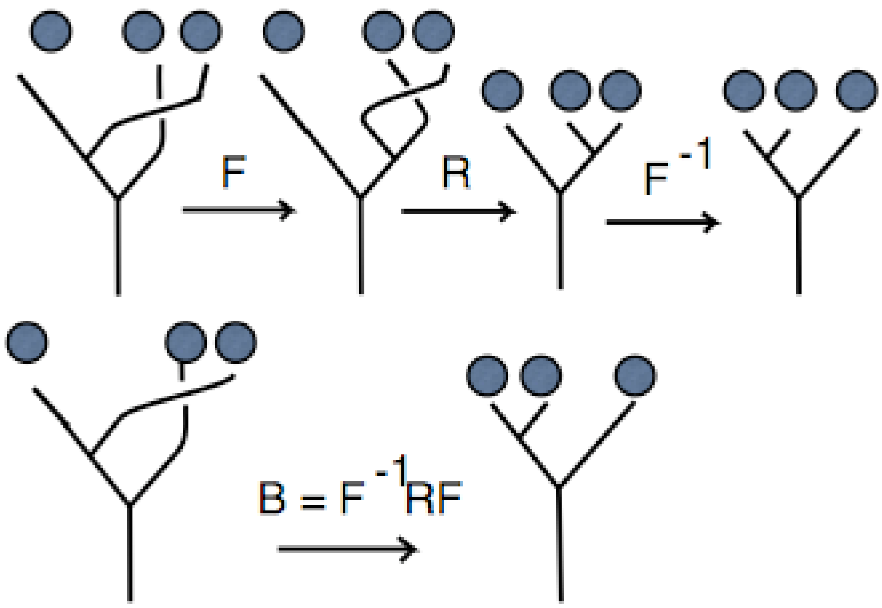
$$R|*\rangle = \mu\, |*\rangle$$

$$R|M\rangle = \lambda\, |M\rangle$$

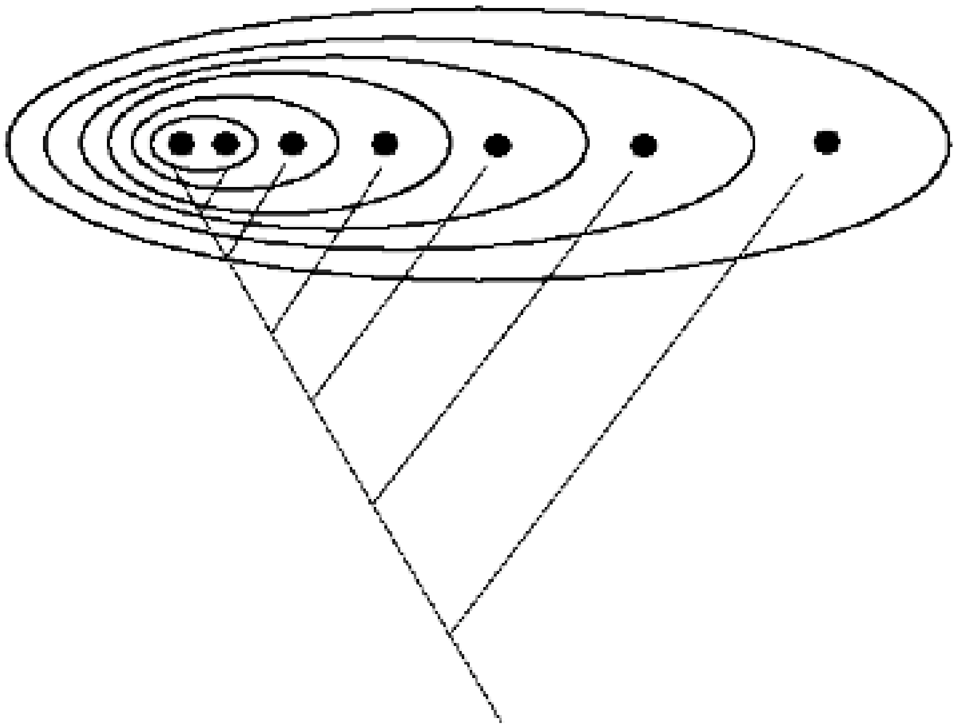
for this phase change, where μ and λ are unit complex numbers. Things get more interesting when we consider braiding of three or more particles.



Then we have to consider changing basis from one process space to another in order to measure the braiding of the second two particles (who have to interact directly to let us see their phase change). This coordinate transformation is here denote by a matrix F that is also 2 x 2 and unitary.



The braiding of the right hand particles is then represented by the matrix $B = F^{-1}RF$ and we have that R and B give a unitary representation of the three-strand braid group. In this way unitary transformations, and hence quantum processes, can be represented by braids. This is the beginning of the theory of topological quantum computing.



In general, the process spaces can be arbitarily large, involving many
particle interactions.

$$A = e^{3\pi i/5}.$$

$$\delta = -A^2 - A^{-2}$$

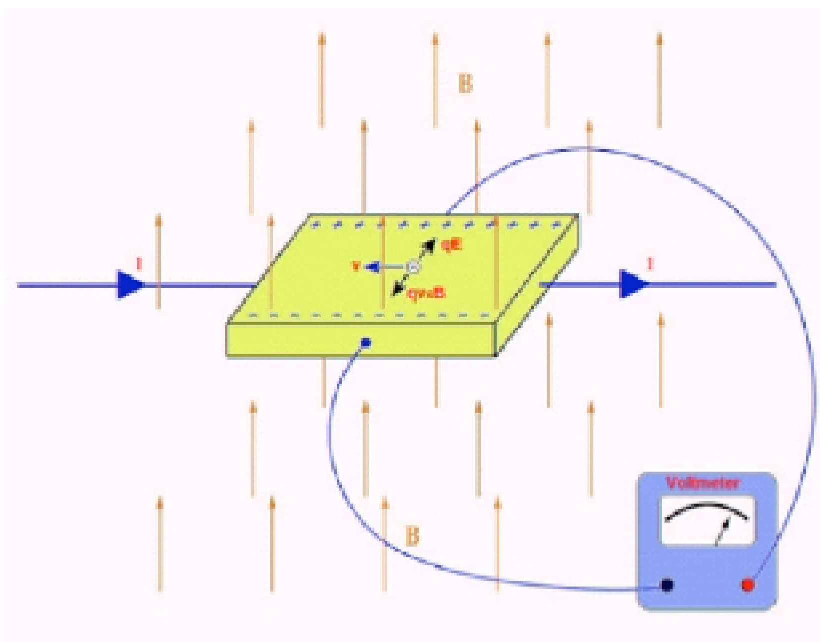
$$\Delta = \delta = (1 + \sqrt{5})/2.$$

$$F = \begin{pmatrix} 1/\Delta & 1/\sqrt{\Delta} \\ 1/\sqrt{\Delta} & -1/\Delta \end{pmatrix} = \begin{pmatrix} \tau & \sqrt{\tau} \\ \sqrt{\tau} & -\tau \end{pmatrix}$$

$$R = \begin{pmatrix} -A^4 & 0 \\ 0 & A^8 \end{pmatrix} = \begin{pmatrix} e^{4\pi i/5} & 0 \\ 0 & -e^{2\pi i/5} \end{pmatrix}.$$

The parameters for R and F shown above give the well-known Fibonacci model. This yields as unitary braid group representation on our process space U and this representation is dense in the set of all unitary 2 x 2 matrices. Thus any quantum process on H can be represented by Fibonacci braiding. The same holds true for the corresponding higher dimensional Fibonacci process spaces. We conclude that such braiding vaults the quantum mechanics of the Fibonaccci processes related to the act of distinction to generate all possible quantum processes.

A possible application of these ideas occurs in the fractional quantum hall effect where collective excitations of electrons in a super-cooled metal plate behave like these Fibonacci particles.



In the quantum Hall effect [MR] the metal plate is in a transverse magnetic field and a current is imposed across the plate.
The resistance of the plate in the direction perpendicular to this current has been observed to be subtly quantized. The braiding of collective electron excitations (quasi-particles) are invoked to explain this effect. It is possible that quantum computers will be built using these effects of braiding.

Acknowledgment.
The author would like to thank Blanches Descartes for many helpful conversations, as it was his privilege to share her office in the academic year 2003 - 2004 while visiting the University of Waterloo and the Perimeter Institute in Waterloo, Canada during which this research was begun.

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