\[ ab = \tilde{x} + a \tilde{b} \]

Basic rule for lifting paths into covering space.

If start with \( G = \langle \tilde{x}_1, \ldots, \tilde{x}_n \rangle \) a free group, then in base space
\[ B = \bigvee_{i=1}^n S^1 = \bigvee \text{ a wedge of circles} \]

cover to give \( \tilde{x}_1, \ldots, \tilde{x}_n \).

\[ p: E \rightarrow B \text{ is a covering space}. \]

\( E \) is an infinite tree, acted on freely by \( G \) with fundamental 1-cells \( \tilde{x}_1, \ldots, \tilde{x}_n \).

\[ n = 2: \]

\[ \begin{array}{c}
\tilde{x}_1 \cdot \tilde{x}_2 \cdot \tilde{x}_1 \cdot \tilde{x}_2 = \tilde{x}_1 + \tilde{x}_1 (\tilde{x}_2 + \tilde{x}_2 (\tilde{x}_1 + \tilde{x}_1 \tilde{x}_2)) \\
= \tilde{x}_1 + \tilde{x}_1 (\tilde{x}_2 + \tilde{x}_2 (\tilde{x}_1 + \tilde{x}_1 \tilde{x}_2)) \\
= \tilde{x}_1 + \tilde{x}_1 \tilde{x}_2 + \tilde{x}_1 \tilde{x}_2 \tilde{x}_1 + \tilde{x}_1 \tilde{x}_2 \tilde{x}_1 \\
= (1 + \tilde{x}_1 \tilde{x}_2) \tilde{x}_1 + (\tilde{x}_1 + \tilde{x}_1 \tilde{x}_2 \tilde{x}_1) \tilde{x}_2
\end{array} \]

Defn. \( G = \langle \tilde{x}_1, \ldots, \tilde{x}_n \rangle \), \( Z[G] = \text{integral group ring of } G \). \( W \in G \) then
\[ \tilde{W} = \frac{\partial W}{\partial \tilde{x}_1} \tilde{x}_1 + \cdots + \frac{\partial W}{\partial \tilde{x}_n} \tilde{x}_n \]
defines operator
\[ D_i = \frac{\partial}{\partial \tilde{x}_i} : G \rightarrow Z[G] \]

If \( D = D_i \) for some \( i \) then
\[ D(ab) = D(a) + \alpha D(b) \]
(follows from \( \tilde{ab} = \tilde{a} + \tilde{a} \tilde{b} \))
and \( D(1) = 0 \), \( \partial \tilde{\chi}_i / \partial \tilde{x}_i = \delta_{ii} \).
These derivatives can then be used to find formulas of lifts of 2-cells in other covering spaces. Let
\[ E \xrightarrow{P} B \] be as above
\[ B' = B \cup \{ \text{2-cells } \sigma_j \text{ } | \text{ } \sigma_j = x_j \in G \} \]
i.e. \( x_j \) = ordered product of group gens in free group \( G \).

Thus \( \pi_1(B') = (x_1, x_2, \ldots, x_n | x_1, x_2, \ldots, x_m) \).

Let \( E' \xrightarrow{P'} B' \) be a regular covering space, correspon to a normal subgroup \( H \trianglelefteq G = \pi_1(B') \).

Then if \( \text{Symm}(E'/B') = \{ f : E' \to E' | P'f = P', \text{ f homom} \} \)

the group of deck transits for \( E' \) then
\[ \text{Symm}(E'/B') = G'/H. \]

Let \( \psi : G = (x_1, x_2, \ldots, x_n) \to G'/H \) be canonical map to this quotient.

Then: If \( \sigma_j \) = lift of the cell \( \sigma_j \) to \( E' \), then
\[ \tilde{\sigma}_j = (\sigma_j) = \left( \frac{\partial \theta_j}{\partial x_1} \right) \tilde{x}_1 + \cdots + \left( \frac{\partial \theta_j}{\partial x_n} \right) \tilde{x}_n \]

where \( \tilde{x}_i = \text{lift of } x_i \text{ into } E' \).

This formula shows how the Jacobian matrix \( \left( \frac{\partial \theta_j}{\partial x_i} \right) \psi \) can be used to compute \( H_1(E') \).

Application to knot theory (and more generally)
\[ H = \left[ G', G' \right] = \left[ \pi_1(B'), \pi_1(B') \right] \]

commutator subgroup.

Then \( G'/H = \text{Abelianization } (G') \).

If we start with \( G' = \pi_1(S^3 - K) \)
where \( K \subset S^3 \) is a knot (1-component)
then \( \psi : G' \to \text{Cox} = (\pi_1) \)
In the case of the Wirtinger presentation we have
\[
\begin{align*}
\alpha b &= c b \quad \text{or} \quad c = b a b^{-1} \\
\alpha b &= b c \quad \text{or} \quad c = b^{-1} a b
\end{align*}
\]

Each meridional generator \(a, b, c, \ldots\) \(\rightarrow \lambda\) under \(\Psi\).

Each relation in \(\pi_1(S^3 - K)\) corresponds to a relation in the \(\Lambda\)-module \(H_1(E_{oo})\) where \(\Lambda = \mathbb{Z}[x, x^{-1}]\) and \(E_{oo} = \text{covering space of } (S^3 - K)\) correponds to \([\pi_1(S^3 - K), \pi_1(S^3 - K)]\).

(i) \(ba = cb \Rightarrow \bar{ba} = \bar{cb}\)
\[
\bar{b} + b \bar{a} = \bar{c} + c \bar{b}
\]
\[
\bar{c} = b \bar{a} + (1-c) \bar{b}
\]

applying \(\Psi\): \(\bar{c} = \alpha \bar{a} + (1-\alpha) \bar{b}\)

(ii) \(ab = bc \Rightarrow \bar{ab} = \bar{bc}\)
\[
\bar{a} + a \bar{b} = \bar{b} + b \bar{a}
\]
\[
b \bar{c} = \bar{a} + (\bar{a} - 1) \bar{b}
\]
\[
\bar{c} = b^{-1} \bar{a} + (b^{-1} \alpha - b^{-1}) \bar{b}
\]

applying \(\Psi\): \(\bar{c} = \alpha^{-1} \bar{a} + (1-\alpha^{-1}) \bar{b}\)

These then give the generators and relations for the \(\Lambda\)-module \(H_1(E_{oo})\).

Using Wirtinger presentation, it is easy to see that the cycles in \(H_1(E_{oo})\) are given by diffs \(\bar{\beta}_i - \bar{\alpha}_i\) (fixing \(\bar{\alpha}_i\)). We have one redundant relation as well.
The upshot is that we get exactly the relation matrix for $H_1(E_\infty)$ by taking out one row and one column from $(\partial x_i/\partial x_j)$. 

\[ \begin{align*}
\text{Ex: } & a & b \\
\downarrow & c & \downarrow \\
\text{c} &= a \bar{b} = \bar{x} a + (1-\bar{x}) b \\
\text{b} &= c \bar{a} = \bar{x} c + (1-\bar{x}) a \\
\text{a} &= b \bar{c} = \bar{x} b + (1-\bar{x}) c \\
\end{align*} \]

(this formalism replaces $\bar{x} = x^2 + (1-x) b + e + c$.)

\[ \begin{pmatrix} \bar{x} a \bar{b} \\ \bar{b} \bar{a} \end{pmatrix} \]

\[ \begin{pmatrix} \bar{x} a \bar{b} \\ \bar{b} \bar{a} \end{pmatrix} \]

\[ \begin{array}{|c|c|c|}
\hline
x & 1-x & -1 \\
\hline
1-x & -1 & x \\
-1 & x & 1-x \\
\hline
\end{array} \]

\[ \Rightarrow \begin{pmatrix} -1 & x \\
1-x & 1-x \end{pmatrix} = \text{reln matrix for } H_1(E_\infty). \]

We can determine the structure of $H_1(E_\infty)$ by doing row and col ops on $J$ over $\Lambda = \mathbb{Z}[x, x^{-1}]$.

\[ \begin{pmatrix} -1 & x \\
1-x & 1-x \end{pmatrix} \xrightarrow{\text{row ops}} \begin{pmatrix} 1 & 0 \\
0 & 1-x + x^2 \end{pmatrix} \xrightarrow{\text{col ops}} \begin{pmatrix} 1 & 0 \\
0 & 1-x + x^2 \end{pmatrix} \]

This shows that $H_1(E_\infty)$ is a cyclic module: $H_1(E_\infty) \cong \Lambda/(1-x + x^2)\Lambda$.

We say that $\Delta_K(x) = x^2 - x + 1$ is the Alexander polynomial of $K$.

More generally, $\Delta_K(x)$ is determined up to $a \cong b$ where $a \cong b$ means $a = \pm x^N b$, and $\Delta_K(x)$ is defined to be the generator (in $\Lambda$) of the ideal generated by $(n-1) \times (n-1)$ minors in $(\partial x_i/\partial x_j)$, where $(x_i \ldots x_k)/\Lambda(x_i, \ldots, x_m) = G'$ and $G'/[G', G'] = C_\infty$. 

It is an interesting fact of life that for knots and links, \( H_1(\mathbb{E}^3) \) can often be understood by using more geometric pictures of the covering space.

\[
K = \emptyset F C F \subset S^3
\]

\( F \) an orientable spanning surface for \( K \subset S^3 \).

\[
X = S^3 - K \quad \text{split along} \quad F.
\]

\[
\exists X = F \cup F_+
\]

\[
E_\infty = \ldots \dot{x}^{-2} X \cup x^1 X \cup x^0 X \cup x^2 X \cup x^3 X \cup \ldots
\]

\[
\ldots \quad \dot{x} X \quad \dot{x} X \quad \dot{x}^2 X \quad \ldots
\]

\[
\begin{align*}
\dot{x} F &= F_+ \\
\ddot{x} F &= F
\end{align*}
\]

Note that \( X = S^3 - \text{Nbhd}(F) \). Thus we need to look at \( H_1(S^3 - F) \).

\[
\text{lk} : H_1(F) \times H_1(S^3 - F) \to \mathbb{Z}
\]

Alexander duality pairing

\[
\text{lk}(\hat{a}, a) = 1 \quad \text{lk}(\hat{a}, b) = 0
\]

\[
\text{et cetera.}
\]

Let \( \hat{z} : F \to S^3 - F \) via push by small amount along positive normal.

Write \( \hat{z}(x) = x^* \).

\[
a^* = \hat{z}(a) = c_1 \hat{a} + d_1 \hat{b}
\]

\[
b^* = \hat{z}(b) = c_2 \hat{a} + d_2 \hat{b}
\]

\[
\Rightarrow \text{lk}(a^*, b) = d_1 \\
\text{lk}(a^*, a) = c_1 \\
\text{lk}(b^*, b) = d_2 \\
\text{lk}(b^*, a) = c_2
\]

\[
H_1(S^3 - F) \cong H_1(F)
\]
Define $\Theta : H_1(F) \times H_1(F) \rightarrow \mathbb{Z}$

$\Theta(x, y) = lk(x^*, y^*)$.

Say $\{a_1, \ldots, a_k\}$ basis for $H_1(F)$

$\{\hat{a}_1, \ldots, \hat{a}_k\}$ basis for $H_1(S^3 - F)$.

$a_i^* = [a_i] = \sum M_{ji} \hat{a}_j$

$\Rightarrow lk(a_i^*, a_j) = M_{ji}$

So $\Theta^T = M = \text{matrix of } \hat{\Theta} : H_1(F) \rightarrow H_1(S^3 - F)$

with respect to the dual bases $\{a_i\}$, $\{\hat{a}_j\}$.

In $E\infty$ we have the bases $\{\hat{a}_j\}$, $\{x\hat{a}_j\}$, ...

Thus in $H_1(E\infty)$:

$\Theta^T x = t \Theta x$.

In other words,

$\Theta^T - t \Theta$

is a relation matrix for $H_1(E\infty)$ over $\Lambda = \mathbb{Z}[t, t^{-1}]$.

Thus we can obtain the structure of $H_1(E\infty)$ from $\Theta^T - t \Theta$

and $\Delta_k(t) = \text{Det} (\Theta^T - t \Theta)$. 
Ex.

\[ \begin{array}{c|cc}
\theta & a & b \\
\hline
a & -1 & 0 \\
b & 1 & -1
\end{array} \]

\[ \begin{array}{cc}
\tilde{a} & \tilde{b}^*
\end{array} \]

\[ \theta = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}, \quad \theta^T = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \]

\[ \theta^T - \tau \theta = \begin{pmatrix} -1 + \tau & -\tau \\ 1 & -1 + \tau \end{pmatrix} \]

\[ |\theta^T - \tau \theta| = 1 - 2\tau + \tau^2 + \tau = \tau^2 - \tau + 1 \]

This recomputes the Alexander polynomial of the trefoil knot.

\[ \begin{array}{cc}
\tilde{a} & \tilde{b}^*
\end{array} \]

\[ \delta F \oplus F = 0 \]

\[ \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \]

\[ \theta^T - \tau \theta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \tau \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \tau & -1 + \tau \end{pmatrix} \]

\[ \Delta \neq 1 \] as expected since \( \delta F = 0 \) and \( H = \alpha \).

But now

\[ K = \delta F' \]

\[ \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \]

\[ \Delta_K = 1 \]

This shows how to construct knots with Alexander polynomial is 1.
Showing $K$ and $K'$ are not the same, but are related by different relations in the quandle polynomial.

\[
\begin{array}{c|cc}
\theta & a & b \\
\hline
a & 0 & 1 \\
b & 2 & 0 \\
\end{array}
\]

$\theta^{-1} \theta = \begin{pmatrix} 0 & 2 & -x \\ 1-2x & 0 \end{pmatrix}$

The Alexander module has two generators:

$\Delta_K = (2-x)(1-2x)$

The above gives examples of knots $K$ and $K'$ with the same Alexander polynomial, but different (non-isomorphic) Alexander modules. We can generalize this example to

$\theta = \begin{pmatrix} N & 1 \\ 2 & 0 \end{pmatrix}$

$\theta^{-1} \theta = \begin{pmatrix} N-xN & 2-x \\ 1-2x & 0 \end{pmatrix}$

$\tau = \begin{pmatrix} -N & 2-x \\ 1-2x & 0 \end{pmatrix}$

Question: Are there Alexander modules all non-isomorphic for $|N| > 3$?
Exercise: Demonstrate this isotopy!

This shows that the knot \( KN \) is a ribbon knot, where this means that:

1. \( K \) in a ribbon knot \( \neq K = \partial D \)
2. \( \partial D \rightarrow \mathbb{R}^3 \) is an immersed disk with ribbon singularity

where a ribbon singularity consists in a transverse intersection of an interior arc of the disk with an arc that goes between two boundary points.

- \( K \) ribbon \( \Rightarrow K = \partial D, \quad D \subset \mathbb{R}^4 \) (upper 4-space).
  
```
  \[ \frac{\partial D}{\partial t} \]  
```

- \( K \) ribbon \( \Rightarrow \Delta K = f(t) f(1/t) \)
  for some polynomial \( f(t) \).

```
\[ \text{Use tubing trick to create spanning surface such that} \]
```

\[ \text{spanning surface} \]
$S^2 \hookrightarrow S^4$

$T \xrightarrow{\alpha} T^*$

$a \overline{a} = b, \quad e \overline{e} = f, \quad \overline{a} \overline{e} = a$

$b \overline{d} = c, \quad \overline{f} \overline{e} = a$

d \overline{e} = e

$\Rightarrow$ need relab $f = d$

$\Rightarrow$ go both above $f$ below

then same group.

$\pi_1(S^3 - T\#T^*)$: $a \overline{a} = b, \quad e \overline{e} = d, \quad \overline{d} \overline{e} = a$

$b \overline{d} = c, \quad c \overline{e} = d$

$\Rightarrow f = d:\quad a \overline{a} = b, \quad e \overline{e} = d, \quad \overline{d} \overline{e} = a$

$d \overline{d} = b, \quad a \overline{e} = d, \quad e \overline{e} = a$

$\overline{d} \overline{e} = c, \quad c \overline{e} = d$

$\Rightarrow a \overline{e} = e \overline{e} = d, \quad c \overline{e} = e \overline{e} = d$

$\Rightarrow$ $a \overline{e} = e \overline{e} = d, \quad c \overline{e} = e \overline{e} = d$

$\Rightarrow a \overline{e} = a$

$\Rightarrow a \overline{e} = e \overline{e}$

To get an $S^2 \hookrightarrow S^4$ with

$\pi_1(S^4 - S^2) \cong \langle a, e | a \overline{e}a = e \overline{e}a \rangle \cong \pi_1(\mathbb{R})$. 

$\Rightarrow$
Two surfaces \( S, S' \subset \mathbb{R}^3 \subset S^3 \) are \( s \)-equivalent if one can be obtained from the other by a combination of ambient isotopy and tubing. Here tubing means cutting out two disjoint \( D^2 \times S^1 \) and replacing by an embedded \( S' \times S^1 \). (Or the reverse.)

**Theorem (See handout).** \( S, S' \subset \mathbb{R}^3 \subset S^3 \) surfaces with boundary \( K, K' \) resp. Assume \( K \cong K' \) (amb. iso.), then \( S \) and \( S' \) are \( s \)-equivalent.

From this we can prove that:

- \( \nabla K = \left| x^{-1} \Theta^T - x \Theta \right| \) is a precise invariant of the link \( K : K \cong K' \Rightarrow \nabla K = \nabla K' \).
- \( \sigma(K) = \text{Signature} (\Theta + \Theta^T) \) is a precise invariant of \( K \).

\[ \Theta = \text{Seifert pairing when the tube is added.} \]
\[ \tilde{\Theta} = \text{Seifert pairing without the tube.} \]

\[ \tilde{\Theta} = \begin{pmatrix} \Theta & b^T & 0 \\ b & 0 & 1 \end{pmatrix} \]

Change of basis for Seifert pairing means that you can do an invertible row op paired with the identical column op.

\[ \left( \begin{array}{ccc} \Theta & b & 0 \\ b^T & N & 0 \\ 0 & 0 & 1 \end{array} \right) \rightarrow \left( \begin{array}{ccc} \Theta & 0 & 0 \\ b^T & 0 & 1 \end{array} \right) \]

For this,

\[ x^T \tilde{\Theta}^T - x \tilde{\Theta} = \begin{pmatrix} x^T \Theta^T - x \Theta \\ -x^T b \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} x b & 0 \\ 0 & x^T \end{pmatrix} \]

\[ \Rightarrow |x^T \tilde{\Theta}^T - x \tilde{\Theta}| = \left| x^T \Theta^T - x \Theta \right| = \begin{pmatrix} 0 & x^{-1} \\ -x^T & 0 \end{pmatrix} \begin{pmatrix} 0 & x^{-1} \\ -x^T & 0 \end{pmatrix} \]

This proves that \( \nabla K \) is precise.
\[
\tilde{\Theta} = \begin{pmatrix}
\Theta & 0 & 0 \\
BT & 0 & 0 \\
0 & 1 & 0 \\
\end{pmatrix}
\]
\[
\tilde{\Theta} + \tilde{\Theta}^T = \begin{pmatrix}
\Theta & b & 0 \\
BT & 0 & 0 \\
0 & 1 & 0 \\
\end{pmatrix}
\]

\[
\tilde{\Theta} \sim_c \begin{pmatrix}
\Theta & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
\end{pmatrix}
\]

Signature (\(\tilde{\Theta}\)) = Signature (\(\Theta\)) + Signature (\(0_{10}\)) = \emptyset.

This proves that \(\text{Signature}(\Theta) = \sigma^*(K)\) is a precise invariant. //

Note also that \(|\tilde{\Theta} + \tilde{\Theta}^T| = -|\Theta + \Theta^T|\).

Abs Value \((1 \Theta + \Theta^T) \overset{\text{Det}(K)}{\Rightarrow} \text{the determinant of the knot } K\) is a precise invariant.

**Skein Identity for \(\nabla K\)**

\[
\Theta_+ \quad \Theta_- \quad \Theta_0
\]

\[
\Theta_+ = \begin{pmatrix}
\mu & \kappa^T \\
\kappa & \Theta_0 \\
\end{pmatrix}
\]

\[
\Theta_- = \begin{pmatrix}
\mu^{-1} & \kappa^T \\
\kappa & \Theta_0 \\
\end{pmatrix}
\]

\[
\Rightarrow \nabla_+ - \nabla_- = [\kappa^{-1} - \kappa] \mu - (\kappa^{-1} - \kappa) (n+1) \nabla_0
\]

\[
\nabla_+ - \nabla_- = (\kappa - \kappa^{-1}) \nabla_0
\]

**Conway Axioms**

\[
\nabla \rightarrow - \nabla_\times = \kappa \nabla \Rightarrow \quad \zeta = \kappa - \kappa^{-1}
\]

\[
\nabla \Theta = 1
\]

We have proved that \(\nabla K\) satisfies the axioms for the Conway "potential function".