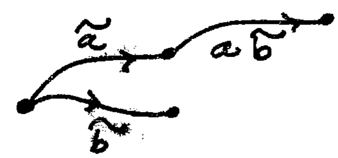
Notes on Alexander Module and Alexander Polynomial

by LK

$\widetilde{ab} = \tilde{a} + a\tilde{b}$



Basic rule for lifting path into covering space.

If start with $G = (x_1, \ldots, x_n \mid \ )$ a free group then use base space

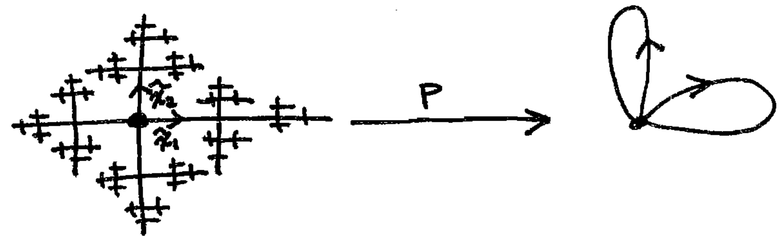
$B = \bigvee_{i=1}^{n} S^1 =$  a wedge of

circles corresp. to gens $x_1, \ldots, x_n$.

$p : E \longrightarrow B$ univ covering space.

E is an infinite tree, acted on freely by G with fundamental 1-cells $\tilde{x}_1, \ldots, \tilde{x}_n$.

$n = 2$:



e.g. $\widetilde{x_1 x_2 x_1 x_2} = \tilde{x}_1 + x_1 (\widetilde{x_2 x_1 x_2})$

$$= \tilde{x}_1 + x_1(\tilde{x}_2 + x_2(\tilde{x}_1 + x_1 \tilde{x}_2))$$

$$= \tilde{x}_1 + x_1 \tilde{x}_2 + x_1 x_2 \tilde{x}_1 + x_1 x_2 x_1 \tilde{x}_2$$

$$= (1 + x_1 x_2)\tilde{x}_1 + (x_1 + x_1 x_2 x_1)\tilde{x}_2$$

<u>Defn.</u> $G = (x_1, \ldots, x_n \mid \ )$, $\mathbb{Z}[G]$ = integral group ring of G. $w \in G$ then

$$\tilde{w} = \frac{\partial w}{\partial x_1}\tilde{x}_1 + \cdots + \frac{\partial w}{\partial x_n}\tilde{x}_n$$

defines operators

$$D_i = \frac{\partial}{\partial x_i} : G \longrightarrow \mathbb{Z}[G].$$

If $D = D_i$ for some i then

$$D(ab) = D(a) + aD(b)$$

(follows from $\widetilde{ab} = \tilde{a} + a\tilde{b}$)

and $D(1) = 0$, $\partial x_i / \partial x_j = \delta_{ij}$.

These derivatives can then be used to find boundaries of lifts of 2 cells in other covering spaces. Let

$E \xrightarrow{p} B$ be as above

$B' = B \cup \{2\text{-cells } \sigma_j\}$ $\quad \partial \sigma_j = r_j \in \mathfrak{G}$

i.e. r_j = ordered product of group gens in free group $\mathfrak{G}$.

Thus $\pi_1(B') = (x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_m)$.

Let $E' \xrightarrow{p'} B'$ be a regular covering space corresp to a normal subgroup $H \triangleleft \mathfrak{G}' = \pi_1(B')$.

Then if $\mathrm{Symm}(E'/B') = \{ f : E' \to E' \mid p'f = p',\ f \text{ homeom} \}$

$$\begin{tikzcd} E' \arrow[rr, "f"] \arrow[dr, "p'"'] & & E' \arrow[dl, "p'"] \\ & B' & \end{tikzcd}$$

(the group of deck transfs for E') then

$\mathrm{Symm}(E'/B') = \mathfrak{G}'/H$. $\quad \boxed{a^{\psi} \underset{\text{def}}{=} \psi(a)}$

Let $\psi : \mathfrak{G} = (x_1, x_2, \ldots, x_n \mid \ ) \longrightarrow \mathfrak{G}'/H$ be canonical map to this quotient.

<u>Then</u>: If $\widetilde{\sigma_j}$ = lift of the cell σ_j to E', then

$$\partial \widetilde{\sigma_j} = \left(\partial \sigma_j\right)^{\psi} = \left(\frac{\partial r_1}{\partial x_1}\right)^{\psi} \widetilde{x}_1 + \cdots + \left(\frac{\partial r_m}{\partial x_n}\right)^{\psi} \widetilde{x}_n$$

where $\widetilde{x}_i$ = lift of x_i into E'.

This formula shows how the Jacobian matrix $\left(\frac{\partial r_i}{\partial x_j}\right)^{\psi}$ can be used to compute $H_1(E')$.

<u>Application to knot theory (& more generally)</u>

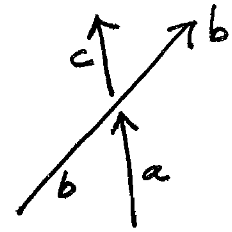
$H = [\mathfrak{G}', \mathfrak{G}'] = [\pi_1(B'), \pi_1(B')]$

commutator subgroup.

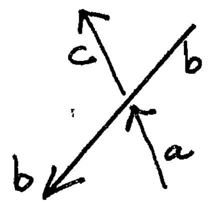
Then $\mathfrak{G}'/H = \text{Abelianization}(\mathfrak{G}')$.

If we start with $\mathfrak{G}' = \pi_1(S^3 - K)$ where $K \subset S^3$ is a knot (1-component) then $\psi : \mathfrak{G}' \longrightarrow C_\infty = (t \mid \ )$

In the case of the Wirtinger presentation we have



$ba = cb$ or $c = bab^{-1}$



$ab = bc$ or $c = b^{-1}ab$

Each meridional generator $a, b, c, \ldots \longmapsto t$ under ψ.

Each relation in $\pi_1(S^3 - K)$ corresponds to a relation in the Λ-module $H_1(E_\infty)$ where $\Lambda = \mathbb{Z}[t, t^{-1}]$ and E_∞ = covering space of $(S^3 - K)$ corres to $[\pi_1(S^3-K), \pi_1(S^3-K)]$.

(1) $ba = cb \Rightarrow \widetilde{ba} = \widetilde{cb}$

$$\tilde{b} + b\tilde{a} = \tilde{c} + c\tilde{b}$$

$$\tilde{c} = b\tilde{a} + (1-c)\tilde{b}$$

applying ψ: $\boxed{\tilde{c} = t\tilde{a} + (1-t)\tilde{b}}$

(2) $ab = bc \Rightarrow \widetilde{ab} = \widetilde{bc}$

$$\tilde{a} + a\tilde{b} = \tilde{b} + b\tilde{c}$$

$$b\tilde{c} = \tilde{a} + (a-1)\tilde{b}$$

$$\tilde{c} = b^{-1}\tilde{a} + (b^{-1}a - b^{-1})\tilde{b}$$

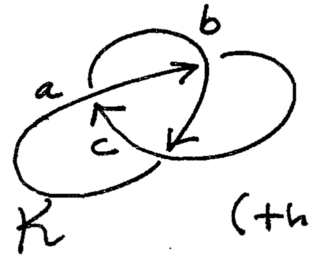
applying ψ: $\boxed{\tilde{c} = t^{-1}\tilde{a} + (1-t^{-1})\tilde{b}}$

These then give the generators and relations for the alex module $H_1(E_\infty)$.

Using Wirtinger presentation, it is easy to see that the cycles in $H_1(E_\infty)$ are gen by diffs $\widetilde{x}_i - \widetilde{x}_1$ (fixing x_1). We have one redundant relation as well.

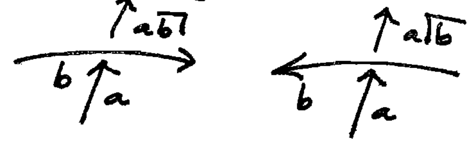
The upshot is that we get exactly the relation matrix for $H_1(E_\infty)$ by <u>taking out one row and one column</u> from $(\partial r_i/\partial x_j)^{\varphi}$.

<u>ex</u>:



$$c = a\overline{b|} = ta + (1-t)b$$
$$b = c\overline{a|} = tc + (1-t)a$$
$$a = b\overline{c|} = tb + (1-t)c$$

(this formalism replaces $\tilde{c} = t\tilde{a} + (1-t)\tilde{b}$ etc.



)

a	b	c
t	$1-t$	-1
$1-t$	-1	t
-1	t	$1-t$

$$\Longrightarrow \begin{pmatrix} -1 & t \\ t & 1-t \end{pmatrix} = J$$ reln matrix for $H_1(E_\infty)$.

We can determine the structure of $H_1(E_\infty)$ by doing row and col ops on J over $\Lambda = \mathbb{Z}[t, t^{-1}]$.

$$\begin{pmatrix} -1 & t \\ t & 1-t \end{pmatrix} \xrightarrow{r} \begin{pmatrix} -1 & t \\ 0 & 1-t+t^2 \end{pmatrix} \xrightarrow{c} \begin{pmatrix} 1 & 0 \\ 0 & 1-t+t^2 \end{pmatrix}$$

This shows that $H_1(E_\infty)$ is a cyclic module: $H_1(E_\infty) \simeq \Lambda/(1-t+t^2)\Lambda$.

We say that $\Delta_K(t) \doteq t^2 - t + 1$ is the <u>Alexander polynomial of K</u>.

More generally, $\Delta_K(t)$ is determined up to $a \doteq b$ where $a \doteq b$ means $a = \pm t^N b$, and $\Delta_K(t)$ is defined to be the generator (in Λ) of the ideal gen by $(n-1)\times(n-1)$ minors in $\left(\frac{\partial r_i}{\partial x_j}\right)^{\varphi}$

where $(x_1, \dots x_n / r_1, \dots, r_m) = G'$ and $G'/[G', G'] = C_\infty$.

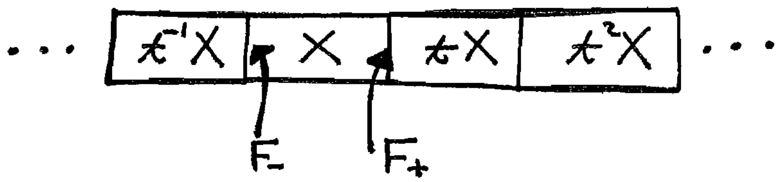
It is an interesting fact of life that for knots and links, $H_1(E_\infty)$ can often be understood by using more geometric pictures of the covering space.

$K = \partial F \subset F \subset S^3$ — F an orientable spanning surface for $K \subset S^3$.

$$X = S^3 - K \ \underline{\text{split along } F}.$$

$$\partial X = F_- \cup F_+$$

$$E_\infty = \cdots t^{-2}X \cup t^{-1}X \cup \mathbf{X} \cup t^2 X \cup t^3 X \cup \cdots$$

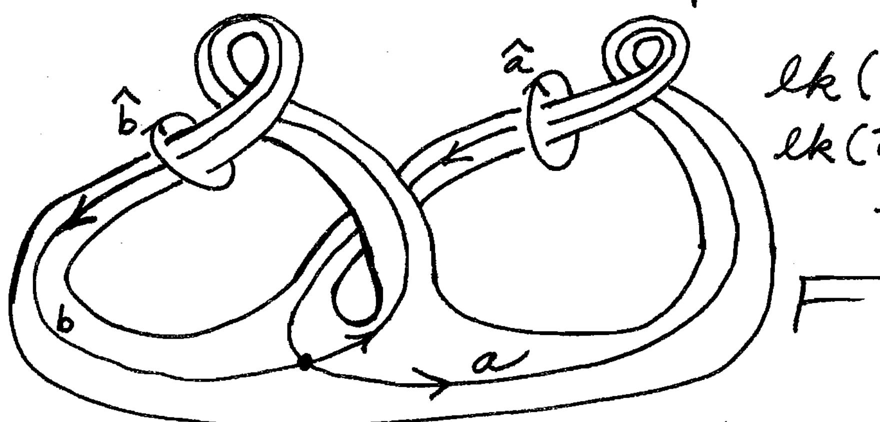


$$\boxed{tF_- = F_+}$$

Note that $X = S^3 - Nbhd(F)$. Thus we need to look at $H_1(S^3 - F)$.

$$lk : H_1(F) \times H_1(S^3 - F) \longrightarrow \mathbb{Z}$$

<u>Alexander duality pairing</u>



$lk(\hat{a}, a) = 1$
$lk(\hat{a}, b) = 0$
<u>etc.</u>

$$\boxed{H_1(S^3 - F) \cong H_1(F)}$$

Let $i : F \longrightarrow S^3 - F$ via push by small amount along positive normal.

Write $i(x) = x^*$.

$a^* = i(a) = c_1 \hat{a} + d_1 \hat{b}$
$b^* = i(b) = c_2 \hat{a} + d_2 \hat{b}$

$\Rightarrow lk(a^*, b) = d_1$
$lk(a^*, a) = c_1$
$lk(b^*, b) = d_2$
$lk(b^*, a) = c_2$

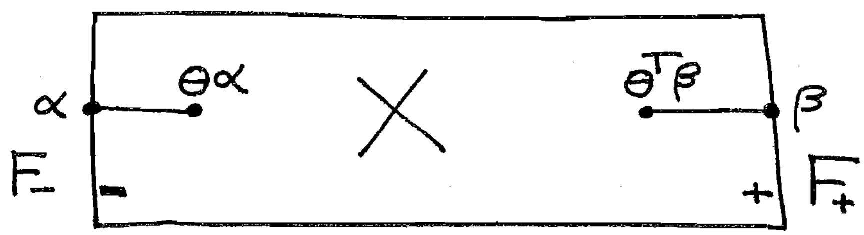
Define $\Theta : H_1(F) \times H_1(F) \longrightarrow \mathbb{Z}$
$$\Theta(x,y) = lk(x^*, y).$$

Say $\{a_1, \dots, a_k\}$ basis for $H_1(F)$
$\{\hat{a}_1, \dots, \hat{a}_k\}$ basis for $H_1(S^3 - F)$.

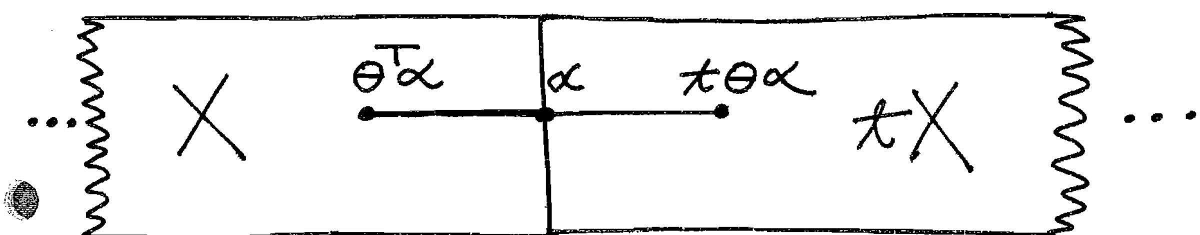
$$a_i^* = i(a_i) = \sum_j M_{ji}\, \hat{a}_j$$

$$\Rightarrow lk(a_i^*, a_j) = M_{ji}$$

So $\Theta^T = M$ = matrix of $i : H_1(F) \to H_1(S^3-F)$ with respect to the dual bases $\{a_i\}$, $\{\hat{a}_j\}$.



In E_∞ we have the bases $\{\hat{a}_i\}, \{t\hat{a}_i\}, \dots$



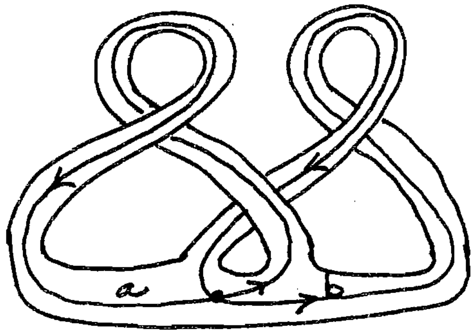
Thus in $H_1(E_\infty)$: $\boxed{\Theta^T\alpha = t\Theta\alpha}$.

In other words,
$$\Theta^T - t\Theta$$
is a relation matrix for $H_1(E_\infty)$ over $\Lambda = \mathbb{Z}[t, t^{-1}]$.

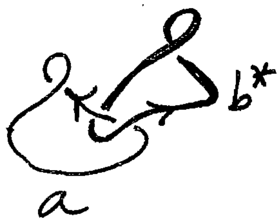
Thus we can obtain the structure of $H_1(E_\infty)$ from $\Theta^T - t\Theta$

and
$$\boxed{\Delta_K(t) \doteq \mathrm{Det}(\Theta^T - t\Theta)}.$$

Ex.



θ	a	b
a	-1	0
b	1	-1

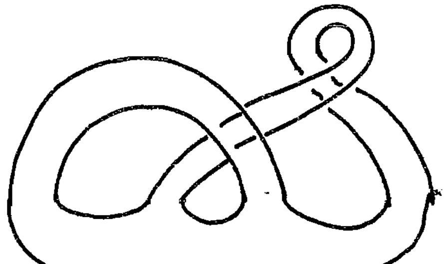


$$\theta = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \quad \theta^T = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$\theta^T - t\theta = \begin{pmatrix} -1+t & -t \\ 1 & -1+t \end{pmatrix}$$

$$|\theta^T - t\theta| = 1 - 2t + t^2 + t = t^2 - t + 1$$

This recomputes the Alex polynomial of the trefoil knot.



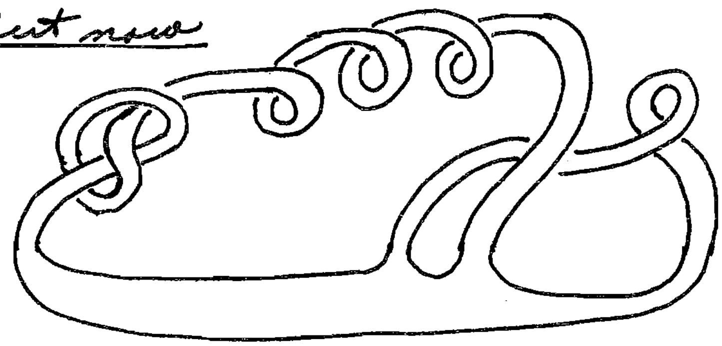
F, $\partial F \simeq \bigcirc$

$$\theta = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

$$\theta^T - t\theta = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} - t\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -t & -1+t \end{pmatrix}$$

$\Delta \doteq 1$ as expected since ∂F = unknot

But now



$K = \partial F'$

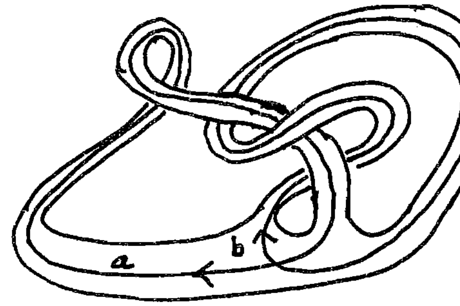
$$\theta = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

$$\Delta_K \doteq 1$$

This shows how to construct knots with Alexander polynomial $= 1$.

Showing K knotted involves using the fundamental group or other knot invariants such as the Jones polynomial.

$K = \partial F$



$K' = \partial F'$

Θ	a	b
a	0	1
b	2	0

Θ'	a	b
a	1	1
b	2	0

$$\Theta^T - t\Theta = \begin{pmatrix} 0 & 2-t \\ 1-2t & 0 \end{pmatrix}$$

Alex module has <u>two</u> generators.

$$\Delta_K \doteq (2-t)(1-2t)$$

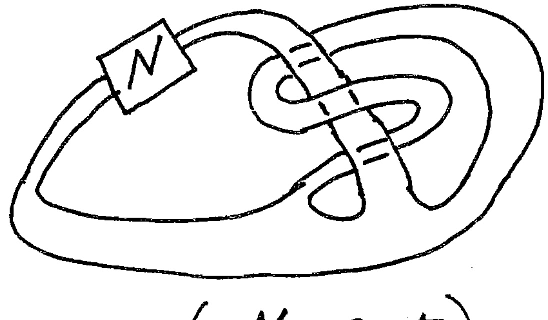
$$\Theta'^T - t\Theta' = \begin{pmatrix} 1-t & 2-t \\ 1-2t & 0 \end{pmatrix} \xrightarrow{c} \begin{pmatrix} -1 & 2-t \\ 1-2t & 0 \end{pmatrix} \xrightarrow{r} \begin{pmatrix} -1 & 2-t \\ 0 & (1-2t)(2-t) \end{pmatrix} \xrightarrow{c} \begin{pmatrix} 1 & 0 \\ 0 & (1-2t)(2-t) \end{pmatrix}$$

Alex module has <u>one</u> generator.

$$\Delta_{K'} \doteq (2-t)(1-2t)$$

The above gives examples of knots K and K' with same Alex polynomial, but <u>different</u> (non-isom) Alexander modules.

We can generalize this example to

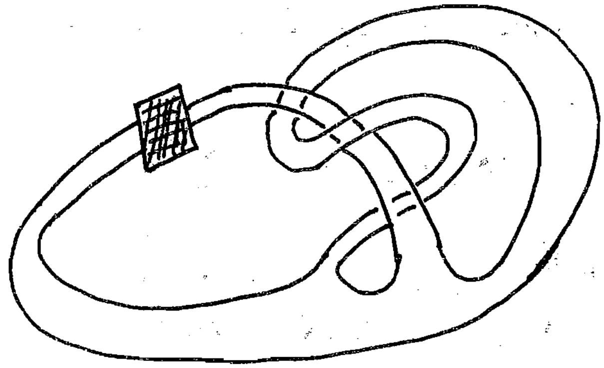


$$\Theta = \begin{pmatrix} N & 1 \\ 2 & 0 \end{pmatrix}$$

$$\Theta^T - t\Theta = \begin{pmatrix} N - tN & 2-t \\ 1-2t & 0 \end{pmatrix}$$

$$\xrightarrow{c} \begin{pmatrix} -N & 2-t \\ 1-2t & 0 \end{pmatrix}$$

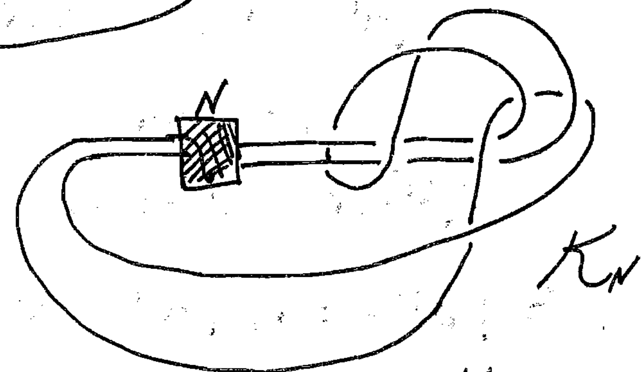
<u>Question</u>. Are these Alex modules all non-isomorphic for $|N| \geq 3$?



Exercise:

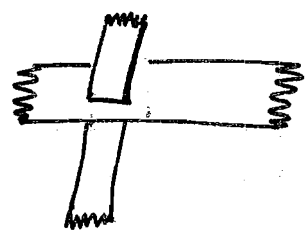
Demonstrate this isotopy!

$\cong$



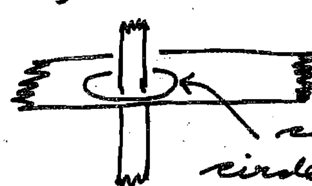
This shows that the knot K_N is a ribbon knot where this means that:

Defn. K is a ribbon knot if $K = \partial D$ where $D \hookrightarrow \mathbb{R}^3$ is an immersed disk with ribbon singularities



where a ribbon singularity consists in a transverse intersection of an interior arc of the disk with an arc that goes between two boundary points.

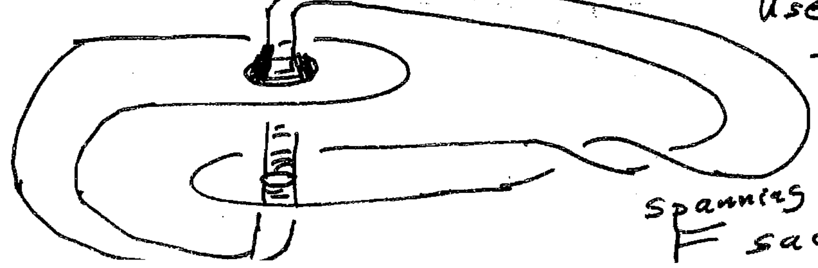
- K ribbon $\Rightarrow$ $K = \partial \mathbb{D}$, $\mathbb{D} \subset \mathbb{R}^4_+$ (upper 4 space). Pf:



cap off this circle with a disk in $\mathbb{R}^4$. //

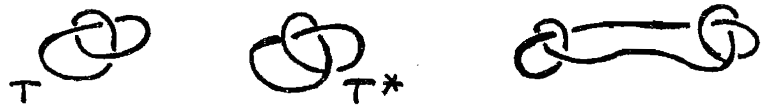
- K ribbon $\Rightarrow$ $\Delta_K \doteq f(t)\, f(1/t)$ for some polynomial $f(t)$.

Pf.

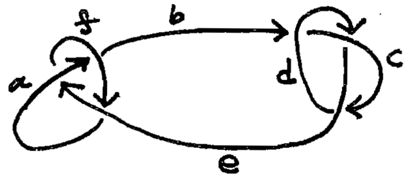


Use tubing trick to create spanning surface F such that

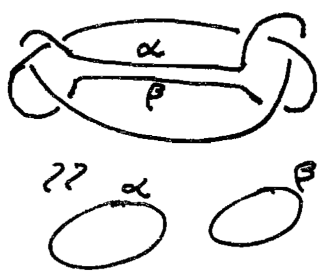
$\boxed{S^2 \hookrightarrow S^4}$



$T \# T^*$



$a\overline{f|} = b$ $\quad e\overline{a|} = f$
$b\overline{|d} = c$ $\quad f\overline{e|} = a$
$c\overline{|e} = d$
$d\overline{|c} = e$



$\Rightarrow$ need relⁿ $f = d$

If go both above & below, then same grp.

$\pi_1(S^3 - T\#T^*)$: $a\overline{f|} = b \quad e\overline{a|} = f \quad f\overline{e|} = a$
$b\overline{|d} = c \quad c\overline{|e} = d \quad \cancel{d\overline{|c} = e}$

Setting $f = d$: $a\overline{d|} = b,\ e\overline{a|} = d,\ d\overline{e|} = a$
$b\overline{|d} = c,\ c\overline{|e} = d$

$da\bar{d} = b,\ ae\bar{a} = d,\ ed\bar{e} = a$
$\bar{d}bd = c \quad \bar{e}ce = d$

$da = bd,\ ae = da,\ \underbrace{ed = ae}$
$bd = dc,\ \underbrace{ce = ed}$ $\Rightarrow ae = ce$
$\Rightarrow \underline{\underline{a = c}}$

$\Downarrow$

$da = bd,\ ae = da,\ \underbrace{ed = ae}_{d = \bar{e}ae}$
$\cancel{bd = da},\ \cancel{ae = ed}$

$\underline{\bar{e}aea} = b\bar{e}ae,\ ae = \underline{\bar{e}aea}$
$\Rightarrow b\bar{e} = 1 \Rightarrow \underline{\underline{b = e}}$

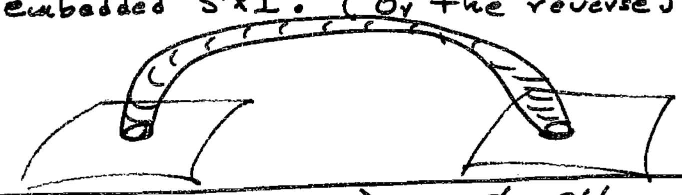
$\Rightarrow \bar{e}aea = e\bar{e}ae,\ ae = \bar{e}aea$

$\Rightarrow \bar{e}aea = ae$
$\Rightarrow \boxed{aea = eae}$

So get an $S^2 \hookrightarrow S^4$ with
$\pi_1(S^4 - S^2) \cong (a, e \mid aea = eae) \cong \pi_1($$)$ //

S-Equivalence

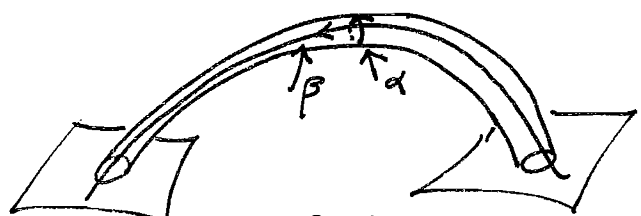
Two surfaces $\subset \mathbb{R}^3 \subset S^3$ are <u>s-equivalent</u> if one can be obtained from the other by a combination of ambient isotopy and <u>tubing</u>. Here tubing means cutting out two disjoint D^2's + replacing by an embedded $S^1 \times I$. (Or the reverse)



<u>Thm</u> (See handout). $S, S' \subset \mathbb{R}^3 \subset S^3$ surfaces with boundary K, K' resp. Assume $K \simeq K'$ (amb. iso.), <u>then</u> S and S' are s-equivalent.

From this we can prove that

- $\nabla_K = |t^{-1}\Theta^T - t\Theta|$ is a precise invariant of the link K: $K \simeq K' \Rightarrow \nabla_K = \nabla_{K'}$.
- $\sigma(K) = \text{Signature}(\Theta + \Theta^T)$ is a precise invariant of K.



$\widetilde{\Theta}$ = Seifert pairing when the tube is added.

Θ = Seifert pairing without the tube.

$$\widetilde{\Theta} = \begin{array}{c|c} & \begin{array}{cc} \beta & \alpha \end{array} \\ \begin{array}{c} \\ \beta \\ \alpha \end{array} & \left(\begin{array}{c|cc} \Theta & b & 0 \\ \hline b^T & N & 0 \\ 0 & 1 & 0 \end{array}\right) \end{array}$$

Change of basis for Seifert pairing means that you can do an invertible row op <u>paired</u> with the identical column op.

$$\left(\begin{array}{c|cc} \Theta & b & 0 \\ \hline b^T & N & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow[r/c]{} \left(\begin{array}{c|cc} \Theta & 0 & 0 \\ \hline b^T & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$$ use <u>this</u> <u>for $\widetilde{\Theta}$</u>

$$t^{-1}\widetilde{\Theta}^T - t\widetilde{\Theta} = \left(\begin{array}{c|cc} t^{-1}\Theta^T - t\Theta & t^{-1}b & 0 \\ \hline -tb^T & 0 & t^{-1} \\ 0 & -t & 0 \end{array}\right)$$

$$\Rightarrow |t^{-1}\widetilde{\Theta}^T - t\widetilde{\Theta}| = |t^{-1}\Theta^T - t\Theta| \underbrace{\begin{vmatrix} 0 & t^{-1} \\ -t & 0 \end{vmatrix}}_{1}$$

This proves that ∇_K is precise.

$$\tilde{\Theta} \equiv \left(\begin{array}{c|cc} \Theta & 0 & 0 \\ \hline b^T & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \qquad \tilde{\Theta} + \tilde{\Theta}^T = \left(\begin{array}{c|cc} \Theta & b & 0 \\ \hline b^T & 0 & 1 \\ 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{r/c} \left(\begin{array}{c|cc} \Theta & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right)$$

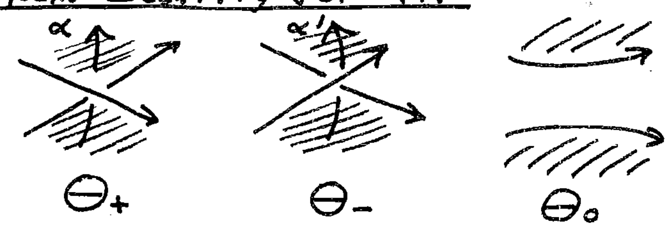
$$\text{Signature}(\tilde{\Theta}) = \text{Signature}(\Theta) + \underbrace{\text{Signature}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{=\phi}.$$

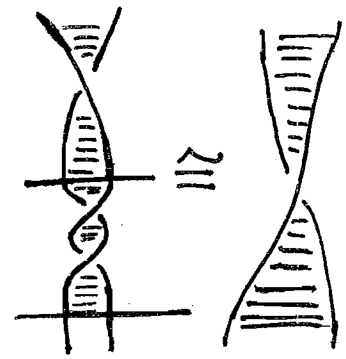
This proves that Signature$(\Theta) = \sigma(K)$ is a precise invariant. //

Note also that $|\tilde{\Theta} + \tilde{\Theta}^T| = -|\Theta + \Theta^T|$.

Abs Value $(|\Theta + \Theta^T|) \underset{def}{=} \underline{\text{Det}(K)}$, the determinant of the knot K, is a precise invariant.

Skein Identity for ∇_K





$$\Theta_+ = \left(\begin{array}{c|c} n & x^T \\ \hline x & \Theta_0 \end{array}\right)$$

$$\Theta_- = \left(\begin{array}{c|c} n+1 & x^T \\ \hline x & \Theta_0 \end{array}\right)$$

Will also discuss
- Arf Invariant
- Knot Cobordism
- Homfly etc...

(state sums)

$$\Rightarrow \nabla_+ - \nabla_- = \left[(t^{-1}-t)n - (t^{-1}-t)(n+1)\right]\nabla_0$$

$$\boxed{\nabla_+ - \nabla_- = (t - t^{-1})\nabla_0}$$

Conway Axioms

$$\nabla_{\times} - \nabla_{\times} = z\nabla_{=}, \quad z = t - t^{-1}$$

$$\nabla_{\bigcirc} = 1$$

We have proved that ∇_K satisfies the axioms for the Conway "potential function".