Here is a picture of the figure eight (B) and its universal covering space E. Now \( \pi_1(B) = \langle x, y \rangle \), the free group on two generators. Let \( G = \langle x, y \rangle \) and note that G is the group of automorphisms of \( E \) over \( B \).

Thus E has “generating” 1-simplices \( X \) and \( Y \) as depicted. \( X \) is the lift of \( x \) as an element in \( \pi_1 \) starting at \( * \). \( Y \) is the lift of \( y \). By regarding \( E \) as the set of translates of \( X \) and \( Y \) under the action of \( G \), we can write the lift of any word \( \omega \in \pi_1(B) \) as a formal sum of simplices with coefficients in \( G \). Thus (letting \( \tilde{\omega} \) denote the lift),

\[
\tilde{\omega} = x^2yx^{-1}\tilde{X} + x^2\tilde{Y} + x^2yx^{-1}\tilde{X}.
\]

Example: \( x^2yx^{-1} = \omega \).

Note that we lift in lexicographic order. Therefore

\[
\tilde{\omega}_1\tilde{\omega}_2 = \tilde{\omega}_1 + \omega_1\tilde{\omega}_2.
\]

The coefficients belong to \( \Gamma = Z[G] \), the group ring over the integers.

The coefficient of \( X \) is called \( \partial \omega / \partial x \).

The coefficient of \( Y \) is called \( \partial \omega / \partial y \).

\[
\tilde{\omega} = \frac{\partial \omega}{\partial x} X + \frac{\partial \omega}{\partial y} Y.
\]
map. Its entries are now in $Z[t, t^{-1}]$. $A_K(t)$ is, up to balance, any generator of the ideal generated by largest minors of $J^\phi$. (This is a principal ideal.)

Let $K_{a,b}$ be a torus knot of type $a,b$. Here $\gcd(a,b) = 1$. We can see that $\pi_1(S^3-K_{a,b}) = \langle a, b | a^a = b^b \rangle$ by looking at $S^3-K_{a,b}$ as the union of pieces interior and exterior to the torus where $K_{a,b}$ lives, and using the Seifert-VanKampen Theorem [NA].

Here is Fox’s algorithm for computing the Alexander polynomial from a presentation of $\pi_1(S^3-K)$. Let $\pi_1(S^3-K) = (x_1, \ldots, x_n | r_1, \ldots, r_m)$ be a presentation.

Regard $r_1, \ldots, r_m$ as elements in the free group generated by $x_1, \ldots, x_n$. Form the Jacobian matrix $J = \left[ \frac{\partial r_i}{\partial x_j} \right]$. Let $\phi : \pi_1(S^3-K) \rightarrow Z = (t | )$ be the Abelianizing map. Let $J^\phi = \left[ \frac{\partial r_i}{\partial x_j} \right]^\phi$ be the image of the Jacobian matrix under the
\[ j^* = [1 + t^b + t^{2b} + \ldots + t^{(a-1)b}, -(1 + t^a + t^{2a} + \ldots + t^{a(b-1)})]. \]

Now
\[ 1 + t^b + t^{2b} + \ldots + t^{(a-1)b} = \frac{t^{ab} - 1}{t^b - 1} \]
\[ 1 + t^a + t^{2a} + \ldots + t^{a(b-1)} = \frac{t^{ab} - 1}{t^a - 1} \]

\[ A(t) = \gcd\left(\frac{t^{ab} - 1}{t^b - 1}, \frac{t^{ab} - 1}{t^a - 1}\right) \]
\[ A(t) = \frac{(t^{ab} - 1)(t - 1)}{(t^{a(b-1)})(t - 1)} \]

This is the Alexander polynomial for the torus knot of type \((a, b)\).

\[ A(t) = \frac{(t^{ab} - 1)(t - 1)}{(t^{a(b-1)})(t - 1)} \]

If \(a\) and \(b\) are both odd then
\[ A(-1) = \frac{(-2)(-2)}{(-2)(-2)} = 1. \]

Hence these knots have vanishing Arf invariant.

Suppose \(a\) is odd and \(b\) is even. Then \(A(-1)\) is indeterminate in this form. So apply L'Hopital's Rule.

\[ A(-1) = \frac{((ab)t^{ab} - 1)(t - 1) + (at^{a(b-1)} - 1)(t^b - 1)}{(at^{a-1})(t^b - 1) + (t^{a-1})(bt^{b-1})} \quad t = -1 \]

\[ = \frac{(-ab)(-2) + 0}{(-2)(-b)}. \]
\[ \therefore A(-1) = a. \]

Thus in this case \(\text{ARF}(K_{a,b}) = 0\) or \(1\) according as

\[ a \equiv 1 \text{ or } 3 \pmod{6}. \]

\[ \text{ARF}(K_{3,2}) = 1 \]
\[ \text{ARF}(K_{5,2}) = 1 \]
\[ \text{ARF}(K_{7,2}) = 0 \]
\[ \text{ARF}(K_{9,2}) = 0 \]
\[ \text{ARF}(K_{11,2}) = 1 \]
\[ \text{ARF}(K_{13,2}) = 1 \]

\[ \ldots \]

A four-fold periodicity.

Exercise 11.1. \(K_{n,2}\) has a spanning surface of form

![Diagram of a torus knot]

(1) Verify this periodicity (above) using topological script.

(2) Calculate \(v_K(t)\) for \(K_{3,4}\) by using the Seifert pairing.

One more remark about Alexander Polynomial and Free Differential Calculus:

We can use the Wirtinger Presentation [F1] for \(v_1(S^3-K)\). This associates one meridional generator to each
arc in the knot diagram, and one relation to each crossing:
\[ c = b^{-1}ab, \quad c = bab^{-1}. \]

Let \( \phi : G = \pi_1(S^3 - K) \to Z = \langle t \mid \rangle. \) Then
\[ \phi(\text{any generator in Wirtinger}) = t. \]
Each relation
\[ c = b^{-1}ab^{-1} = w \]
gives rise to a relation in \( H_1(X_\omega) \) of the form
\[ [c] = \left[ \frac{\partial \omega}{\partial a} \right] [a] + \left[ \frac{\partial \omega}{\partial b} \right] [b]. \]
Since \( \left[ \frac{\partial \omega}{\partial c} \right] = 1 \)
this relation corresponds to a row in the Jacobian matrix for Fox's algorithm. In the case of this presentation, the determinant of any \((n-1) \times (n-1)\) minor will produce the Alexander polynomial. (The knot has \( n \) crossings.)

Exercise 11.2. a) Use the notation of the above remark and show that if \( w = b^{-1}ab \) then \( \left[ \frac{\partial \omega}{\partial a} \right]^\phi = (1-t^{-1}) \) and
\[ \left[ \frac{\partial \omega}{\partial b} \right]^\phi = t^{-1}, \]
while if \( w = bab^{-1} \) then \( \left[ \frac{\partial \omega}{\partial a} \right]^\phi = (1-t) \) and
\[ \left[ \frac{\partial \omega}{\partial b} \right]^\phi = t. \]

b) Choose a knot and calculate its Alexander polynomial using the Wirtinger presentation.

c) Part a) of this exercise shows that if \([a], [b], [c] \) connote elements in \( H_1(X_\omega) \) that correspond to lifts of the elements \( a, b, c \in \pi_1(S^3 - K) \), then the following relations ensue in \( H_1(X_\omega) \) as a \( \mathbb{Z}[t, t^{-1}] \) module:

This part of the exercise asks you to compare these patterns with the patterns that arise from trying to represent the fundamental group as a group of affine transformations of the complex plane [DR]: Let
\[ \mathcal{U} = \{ T : \mathbb{C} \to \mathbb{C} \mid T(z) = az + b, \text{ where } a \text{ and } b \]
are elements of \( \mathbb{C} \}. \]

Call this the affine group. Let \( G = \pi_1(S^3 - K) \) with the Wirtinger presentation.

(i) Let \([a, b] \) denote \( T(z) = az + b \). Show that
\[ [a, b][\gamma, \delta][a, b] = [a\gamma, a\delta + b] \]
where \([ \ ][ \ ]\) denotes composition of maps.

(ii) Suppose \( \phi : G \to \mathcal{U} \) is a homomorphism of groups. Show that \( \phi(a) = [t, \psi(a)], \ t \in \mathbb{C}, \)
for a fixed \( t \) independent of the choice of \( a \), given that \( \mathbb{L}(a, K) = +1 \). Hence this holds for all the Wirtinger generators. Given an element \( a \) with \( \mathbb{L}(a, K) = +1 \), let \( \phi(a) = \psi(a). \) Thus \( \phi(a) = [t, \psi(a)] \).