1. Show that the identification space indicated below gives a torus paved by 7 hexagons such that each hexagon shares edges with each of the other 6 hexagons.

2. Using the Penrose coloring formula for plane cubic graphs:
   \[
   [X] = []([]) - [X]
   \]
   \[
   [06] = 3[6]
   \]
   prove that \[
   [\text{Hexagon}] = [\text{Trapezoid}] + [\text{Cyclic}].
   \]

Find an analogous formula for
3. **Brussels Sprouts**

- Start with a number of crosses: +
- A moves: + + \( \rightarrow \) + +

(The connection sprouts a new cross. The new edge does not intersect any previous edges.)

- Sample game:

\[
+ \quad + \quad \rightarrow \quad + \quad {\text{A}} \quad \rightarrow \quad B \quad + \quad +
\]

\[
+ \quad B \quad \rightarrow \quad + \quad A \quad \rightarrow \quad + \quad B
\]

\[
A \quad \rightarrow \quad \rightarrow \ldots
\]

(A) Show that a game with \( n \) crosses will end in a finite number of steps.

(B) Show that a game with \( n \) crosses will have \( F(n) \) moves.
Find the function \( F(n) \).

(C) In the above, you analyzed Brussels sprouts for games in the plane. Generalize your results to
(a) orientable surfaces of genus \( g \).
(b) the projective plane.
4. A cubic graph (3 edges per node) is said to be edge 3-colored if each edge receives one of three colors \( (r, b, p) \) and every vertex sees three distinct colors. For example, \( r \Box b \Box p \) is edge 3-colored.

Consider two-color circuits in such a coloring. They can be of the form \( rb, rp \) or \( bp \). For example, 

\( r \Box b \Box \) is a \( rb \) circuit, \( r \Box p \Box \) is an \( rp \) circuit, and \( b \Box p \Box \) is a \( bp \) circuit.

Define the parity \( \pi(G, \alpha) \) where \( \alpha \) denotes an edge 3-coloring of \( G \) to be the parity (0 if even, 1 if odd) of the sum

\[
S(G, \alpha) = \# (rb \text{ circuits}) + \# (rp \text{ circuits}) + \# (bp \text{ circuits}).
\]

Thus \( \pi(r \Box b \Box p) = 1 \) (odd) since

\[
S(r \Box b \Box p) = 3.
\]

Find two edge 3-colorings for the graph below that have opposite parity.
5. (a) Construct an embedding (tight) of $K_6$ in the torus.

(b) Prove that for a planar drawing of $K_6$ with crossings one needs $\lambda = 9$ for the $\lambda$-cage method to produce an embedding of $K_6$ in the torus.

(c) Find a planar drawing of $K_6$ with crossings and $\lambda = 9$.

6. Recall the definition of parity $\pi(G, \alpha)$ for $\alpha$ an edge 3-coloring of $G$. Let $G$ be any single cubic graph $G$. Let $\alpha$ be any single 3-color circuit for $(G, \alpha)$. Let $\alpha'$ be the coloring of $G$ obtained by switching the colors (e.g., $r \rightarrow b \rightarrow y \rightarrow c$) on $G$.

Claim $\pi(G, \alpha) = \pi(G, \alpha')$ whenever $G$ is planar.

Investigate this claim with examples. Prove it if you can.

7. Prove that for $n \geq 3$, $n$ odd, $J^w$ is not edge 3-colorable.