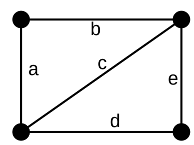
Math 423, Spring 2010, Final Exam

1. A graph is said to be *cubic* if every node has degree three. Let G be a finite connected loop-free cubic plane graph. Show that G must have at least one face with either two, three, four or five edges. (You should use the Euler formula for plane graphs, coupled with the fact that $3v = 2e$ (why?) and the equations that come from counting edges around faces. Please use the notation f_n for the number of faces in G with n edges.)

Give an example of a (non-finite) connected cubic plane graph that has no regions with fewer than six edges. Give an example of a connected cubic plane graph such that all the faces have five edges. Give an example of a connected cubic plane graph such that all the faces have three edges.

2. Let G be the graph shown below.

$G =$ 

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Using the edge-labeling shown above, determine all the spanning trees in G by using both the Wang Algebra and the Kirchoff Matrix.

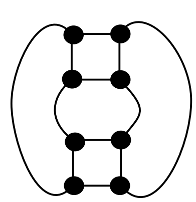
3. Let G be a finite graph with n nodes. Let $A(G)$ be the $n \times n$ adjacency matrix for G. That is, $A(G)_{i,i}$ is the number of loops at the node i and $A(G)_{i,j}$ is the number of edges between node i and node j when $i \neq j$.

Prove that the i, j entry of $A(G)^k$ is equal to the number of walks in G of length k from node i to node j.

Using the characteristic polynomial of $A(G)$, state a recursion relation that can be used to compute the number of walks from vertex i to vertex j of the graph G.

Use these results about the adjacency matrix to tell everything that you can about the walks on G where G is the graph in Problem 2 of this exam.

4. State the definition and the basic properties of the Penrose formula $[G]$ for cubic graphs G immersed in the plane. Use the Penrose formula to determine the number of edge three-colorings of the graph shown below.



5. In this problem, colorability refers to edge three-colorings of cubic graphs (not necessarily planar). Give an example of a connected, isthmus-free cubic graph that is uncolorable. Prove that your example is uncolorable.